Solving Infinite-Domain CSPs Using the Patchwork Property

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- As a consequence, we obtain algorithms running in $f(w) \cdot O(n)$ time for CSPs over Allen's Interval Algebra, RCC8, etc.
- Connecting patchwork to *amalgamation*, we obtain results for temporal constraint satisfaction and phylogeny problems.

Constraint Languages

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- a domain D,
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The relations $\{R_1, R_2, \ldots, R_m\}$ are:

- *jointly exhaustive* (JE) if $\bigcup_{i=1}^{m} R_i = D^k$,
- *pairwise disjoint* (PD) if $R_i \cap R_j = \varnothing$.

Constraint Satisfaction Problem (CSP)

CSP($\mathcal{B}$)

INSTANCE: (V, C), V - variables, C - constraints of form $R(v_1, \dots, v_r)$, where R is a relation from $\mathcal{B}$ and $v_1, \dots, v_r \in V$.

QUESTION: Is there an assignment f of values from to the domain of $\mathcal{B}$ to the variables in V such that $(f(v_1), \dots, f(v_r)) \in R$ for all constraints in C?

Example: CSP over Point Algebra

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Solution: $f(x) = 1, f(y) = f(w) = 2, f(z) = 3$.

Constructing Constraint Languages

Let $\mathcal{B}$ be a k-ary constraint language with JEPD relations.

$\mathcal{B}^{\vee=}$ contains unions of all subsets of relations in $\mathcal{B}$.

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$\langle \mathcal{B} \rangle_{\mathrm{b}}$ contains all relations definable using $\mathcal{B}$-formulas, i.e. logical formulas consisting of the relations in $\mathcal{B}$ and symbols $(,), \wedge, \vee, \neg$.

Example: $\langle (\mathbb{R}; <, =, >) \rangle_{\mathrm{b}}$ contains the relation R_{between} defined as $\{(x, y, z) \in \mathbb{R}^3 \mid (x < y \wedge y < z) \vee (x > y \wedge y > z)\}$.

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$$\mathcal{B} \subsetneq \mathcal{B}^{\vee=} \subsetneq \langle \mathcal{B} \rangle_{\mathrm{b}}.$$

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Let $\mathcal{B}$ be a constraint language. Consider a finite $\Gamma \subseteq \langle \mathcal{B} \rangle_{\mathrm{b}}$.

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If $\mathcal{B}$ has an *infinite* domain, then we cannot enumerate all possible assignments in finite time.

However, if $\mathcal{B}$ is k-ary and has JEPD relations, then we can enumerate all **complete certificates**, i.e. all satisfiable instances of CSP($\mathcal{B}$) with constraints over all k-tuples of variables in V.

Certificates: Example

Let $\Gamma \subseteq \langle(\mathbb{R}; <, =, >)\rangle_{\mathrm{b}}$ be a constraint language. An instance of CSP(Γ) with 3 variables x, y, z has 13 complete certificates:

$$x = y, \quad x = z, \quad y = z$$

$$x = y, \quad x < z, \quad y < z$$

$$x = y, \quad x > z, \quad y > z$$

$$x < y, \quad x = z, \quad y > z$$

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Parameterized Complexity

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A problem is in XP if it admits an algorithm of form $O(n^{f(p)})$.
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We seek fpt algorithms for infinite-domain CSPs parameterized by the *primal treewidth*.

Primal Graph

A *primal graph* associated with an instance (V, C) of CSP has variables V as vertices and an edge for every pair u, v iff u and v appear in the scope of a constraint in C.

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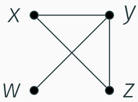
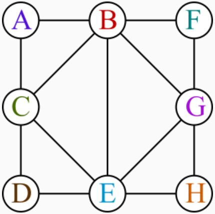
Example: Primal graph of the instance (V, C) of $\mathrm{CSP}(\mathbb{R}; <, =, >)$, where $V = \{x, y, z, w\}$ and $C = \{x < y, y < z, x < z, y = w\}$, is



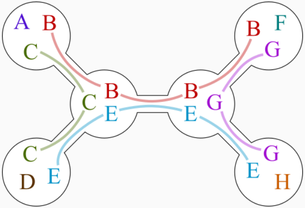
Treewidth

A tree decomposition of a graph G is a tree T and a mapping $X : T \rightarrow 2^V$ such that:

1. If $(u, v) \in E(G)$, then there is $t \in V(T)$ such that $u, v \in X(t)$.
2. For every $v \in V(G)$, the nodes t such that $v \in X(t)$ induce a non-empty connected subtree.



Graph G.

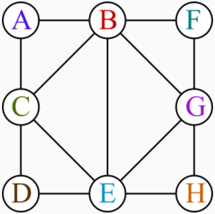


A tree decomposition of G.

Image Credit: David Eppstein

A tree decomposition of a graph G is a tree T and a mapping $X : T \to 2^V$ that satisfies conditions 1 & 2.

Width of (T, X) is the size of the largest $X(t)$ minus one.
Treewidth of G is the minimum width of a tree decomposition of G.
Primal treewidth is the treewidth of the primal graph.



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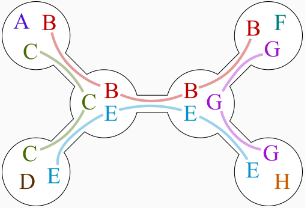

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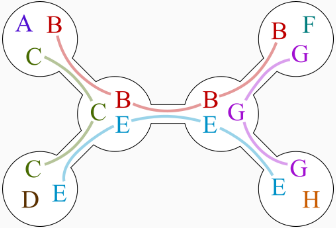
Patchwork: Motivation

Proposition: Finite-domain CSPs are solvable in $f(w) \cdot n^{O(1)}$ time.



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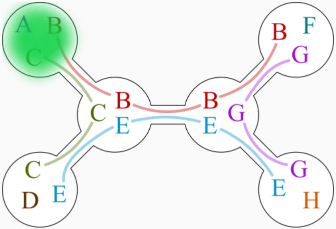

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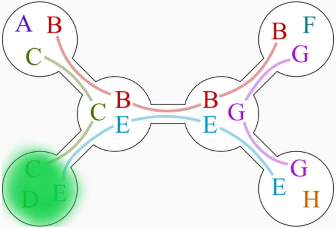

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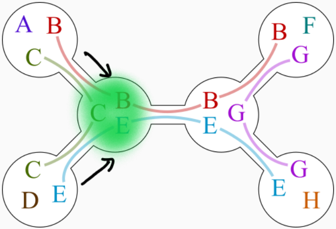

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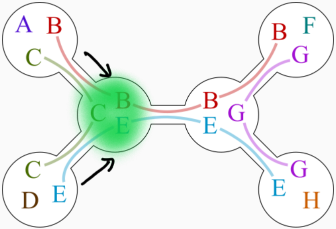

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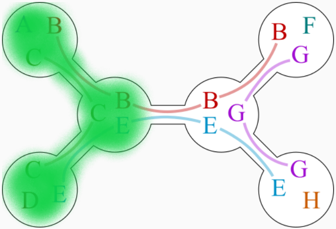

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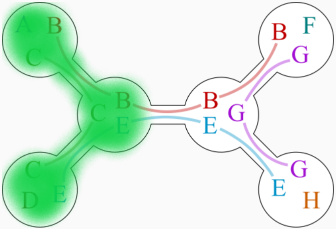

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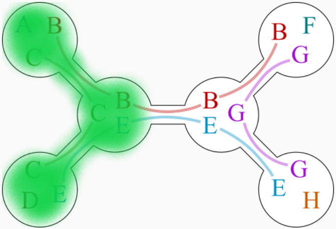

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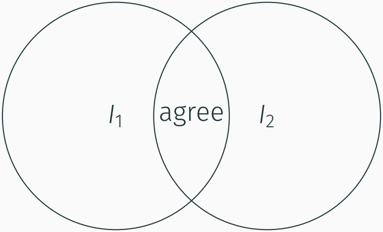
A: Enumerate certificates instead.

Q: How to make step 2 sound?
A: Require the **patchwork property**.

Patchwork: Definition

Definition: A JEPD constraint language $\mathcal{B}$ has *patchwork property* if for every pair of complete satisfiable instances $I_1 = (V_1, C_1)$ and $I_2 = (V_2, C_2)$ of CSP($\mathcal{B}$) such that $I_1[V_1 \cap V_2] = I_2[V_1 \cap V_2]$, the instance $(V_1 \cup V_2, C_1 \cup C_2)$ is also satisfiable.



Fixed-Parameter Tractability

Theorem: Let $\mathcal{B}$ be a finite k-ary constraint language with JEPD relations and the patchwork property. Assume CSP($\mathcal{B}$) is decidable. For any finite constraint language $\Gamma \subseteq \langle \mathcal{B} \rangle_{\mathrm{b}}$, CSP($\Gamma$) is fpt.

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More specifically, an instance of CSP(Γ) is solvable in

$$\tau_{\mathcal{B}}(w+1)^2 \cdot w^k \cdot O(n)$$

time, where $\tau_{\mathcal{B}}$ is the time complexity of enumerating complete instances of CSP($\mathcal{B}$).

Consequences

Point Algebra $(\mathbb{R}; <, =, >)$ has patchwork property.
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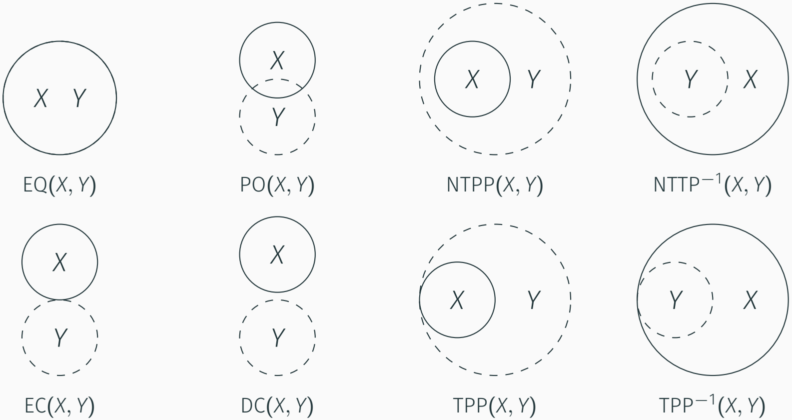
Consequences: Allen's Interval Algebra

Basic relation		Example	Endpoints
I precedes J	p	iii	$I^+ < J^-$
J preceded by I	p^{-1}	jjj	
I meets J	m	$iiii$	$I^+ = J^-$
J met-by I	m^{-1}	$jjjj$	
I overlaps J	o	$iiii$	$I^- < J^- < I^+$,
J overl.-by I	o^{-1}	$jjjj$	$I^+ < J^+$
I during J	d	iii	$I^- > J^-$,
J includes I	d^{-1}	$jjjjjjj$	$I^+ < J^+$
I starts J	s	iii	$I^- = J^-$,
J started by I	s^{-1}	$jjjjjjj$	$I^+ < J^+$
I finishes J	f	iii	$I^+ = J^+$,
J finished by I	f^-1	$jjjjjjj$	$I^- > J^-$
I equals J	e	$iiii$	$I^- = J^-$,
		$jjjj$	$I^+ = J^+$

$\mathrm{AIA} \subseteq \langle(\mathbb{R}; <)\rangle_{\mathrm{fo}} \implies$ CSP($\langle\mathrm{AIA}\rangle_{\mathrm{b}}$) is solvable in $2^{O(w \log w)} \cdot O(n)$.

Consequences: RCC8



$\mathrm{RCC8}$ has patchwork $\implies$ CSP($\langle \mathrm{RCC8} \rangle_{\mathrm{b}}$) is solvable in $2^{O(w^2)} \cdot O(n)$.

Model-Theoretic Point of View

CSP can be thought of a *homomorphism* problem between *relational structures*. A homomorphism is a mapping $h : \mathcal{A} \to \mathcal{B}$ that preserves relations, i.e. if $(a_1, \ldots, a_r) \in R^{\mathcal{A}}$, then $(h(a_1), \ldots, h(a_r)) \in R^{\mathcal{B}}$.

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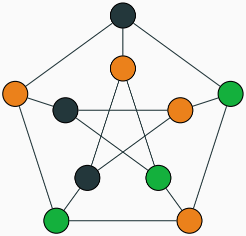
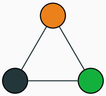
For example, $\mathrm{CSP}(\{0, 1, 2\}; \{\neq\})$ (aka Graph 3-Coloring) asks whether there is a homomorphism from an input graph G to K_3.





Model-Theoretic Point of View

From the model-theoretic point of view, $\mathcal{B}$ has patchwork if it has the *amalgamation property* (AP).

Theorem: If $\mathcal{B}$ is *homogeneous*, then it has AP.

Homogeneity has been verified for many relational structures.

Model-Theoretic Point of View

ω-categoricity is a more general property than homogeneity.

Bodisky & Dalmau have shown that CSP(Γ) is in XP if $\Gamma \subseteq \langle \mathcal{B} \rangle_{\mathrm{b}}$ and $\mathcal{B}$ is ω-categorical, i.e. solvable in $n^{f(w)}$ time for some computable f.

Question: Is there an ω-categorical relational structure $\mathcal{B}$ such that CSP($\mathcal{B}$) is in XP but not in FPT (under plausible complexity-theoretic assumptions)?

Thank you!