Parameterized Approximability of Modular Linear Equations

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Parameterized Approximability of Modular Linear Equations

Joint work with

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- Sebastian Ordyniak (University of Leeds, UK)
- **George Osipov** (University of Oxford, UK)
- Magnus Wahlström (Royal Holloway, University of London, UK)

Systems of Linear Equations over a Ring

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \\ -2y + z &= 2 \\ 2y + w &= 2 \\ 2z + w &= 4 \end{aligned}$$

Is it satisfiable?

Solvable in polynomial time over

- the rational, i.e. $\mathbb{Q}$,
- any finite field, i.e. $\mathbb{F}_p$,
- the integers, i.e. $\mathbb{Z}$,
- the integers modulo any m, i.e. $\mathbb{Z}_m$, …

Systems of Linear Equations over a Ring

Optimization Problem

Min-d-Lin(R)

Instance: $(X, \mathscr{E})$, where
X is the variable set,
$\mathscr{E}$ is a (multi)set of equations, each with at most d variables.

Goal: Find an assignment $\alpha : X \to R$ of minimum cost.

(cost of α is the number of violated equations in $\mathscr{E}$)

Systems of Linear Equations over a Ring

Classical Complexity

Min-2-Lin(R) for any nontrivial ring R (e.g., $\mathbb{Z}_2$) is

- NP-hard,
- UGC-hard to approximate within *any* constant factor.

Min-3-Lin(R) is NP-hard to approximate within *any* constant factor (UGC-free).

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What about parameterized complexity?

Systems of Linear Equations over a Ring

Parameterized Setting

Min-d-Lin(R)

Instance: $(X, \mathscr{E}, k)$

Parameter: k

Decide: is there $\alpha : X \to R$ of cost at most k?

Equivalently: can we delete k equations from $\mathscr{E}$ to make it satisfiable?

Systems of Linear Equations over a Ring

Parameterized Complexity

W[1]-hard cases:

- $d = 3$ [CGJY'13, DJO**O**W'23]
- $d = 2, R = \mathbb{Z}_6$ [DJO**O**W'23]

FPT for $d = 2$ and

- $R = \mathbb{Z}_2$ [RSV'04, CGJY'13]
- $R = \mathbb{Z}_p$ [CCHPP'16]
- $R = \mathbb{Q}, R = \mathbb{Z}$ [DJO**O**W'23]

Open for $d = 2$

- $R = \mathbb{Z}_4$
- $R = \mathbb{Z}_8$
- $R = \mathbb{Z}_9$
- …
- $R = \mathbb{Z}_{p^n}, n \geq 2$

RSV'04 [Reed, Smith, Vetta *OPL'04*]
CGJY'13 [Crowston, Gutin, Jones, Yeo *TOCS'13*]
CCHPP'16 [Chitnis, Cygan, Hajiaghayi, Pilipczuk, Pilipczuk *SICOMP'16*]
DJOOW'23 [Dabrowsi, Jonsson, Ordyniak, Osipov, Wahlström *SODA'23*]

Modular Linear Equations
Parameterized Complexity

Min-2-Lin($\mathbb{Z}_m$)

- FPT if m is a prime ($\mathbb{Z}_m$ is a field)
- W[1]-hard if m has ≥ 2 prime factors ($\mathbb{Z}_m$ is a product ring)
- FPT status open when $m = p^n$, $n \geq 2$

Modular Linear Equations

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Can we at least approximate Min-2-Lin($\mathbb{Z}_m$) in FPT time?

Our Results

Theorem: Min-2-Lin($\mathbb{Z}_{p^n}$) is 2-approximable in FPT time.

Theorem: Min-2-Lin($\mathbb{Z}_m$) is $2\omega(m)$-approximable in FPT time.

* $\omega(m)$ is the number of distinct prime factors of m.

E.g., Min-2-Lin($\mathbb{Z}_{36}$) is 4-approximable in FPT time.

FPT Approximation

Warm-Up: Algorithm for Fields

An FPT 2-approximation for Min-2-Lin$(\mathbb{Z}_5)$:

1) Iterative compression, branching and homogenization reduce to instances where every equation is of the form:

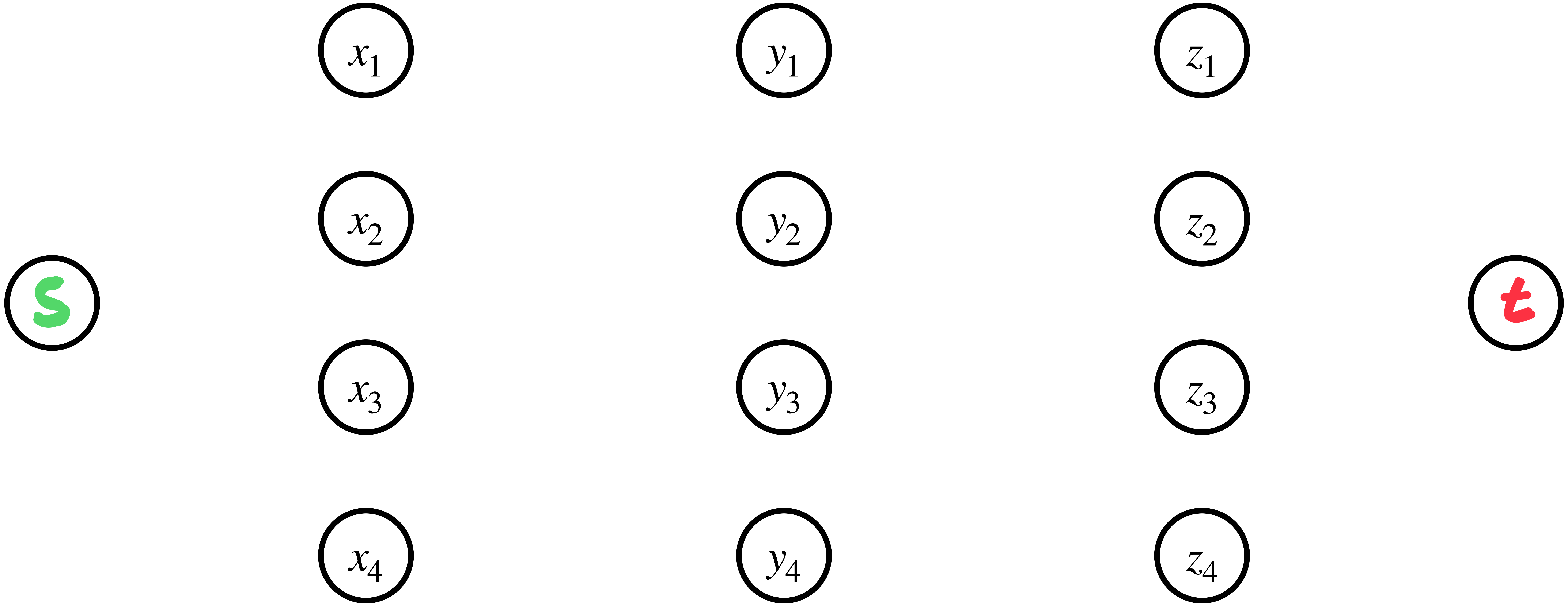
- $x = a$ (for some variable x and value $a \in \mathbb{Z}_5$) or
- $ax = by$ (for some variables x, y and values $a, b \in \mathbb{Z}_5$)

2) Formulate as a graph cut problem and solve it (approximately).

FPT Approximation

Warm-Up: Algorithm for Fields

$$X = \{x, y, z\} \qquad \mathscr{E} = \{x = 2,\ z = 0,\ 2x = y,\ 3y = 2z\}$$



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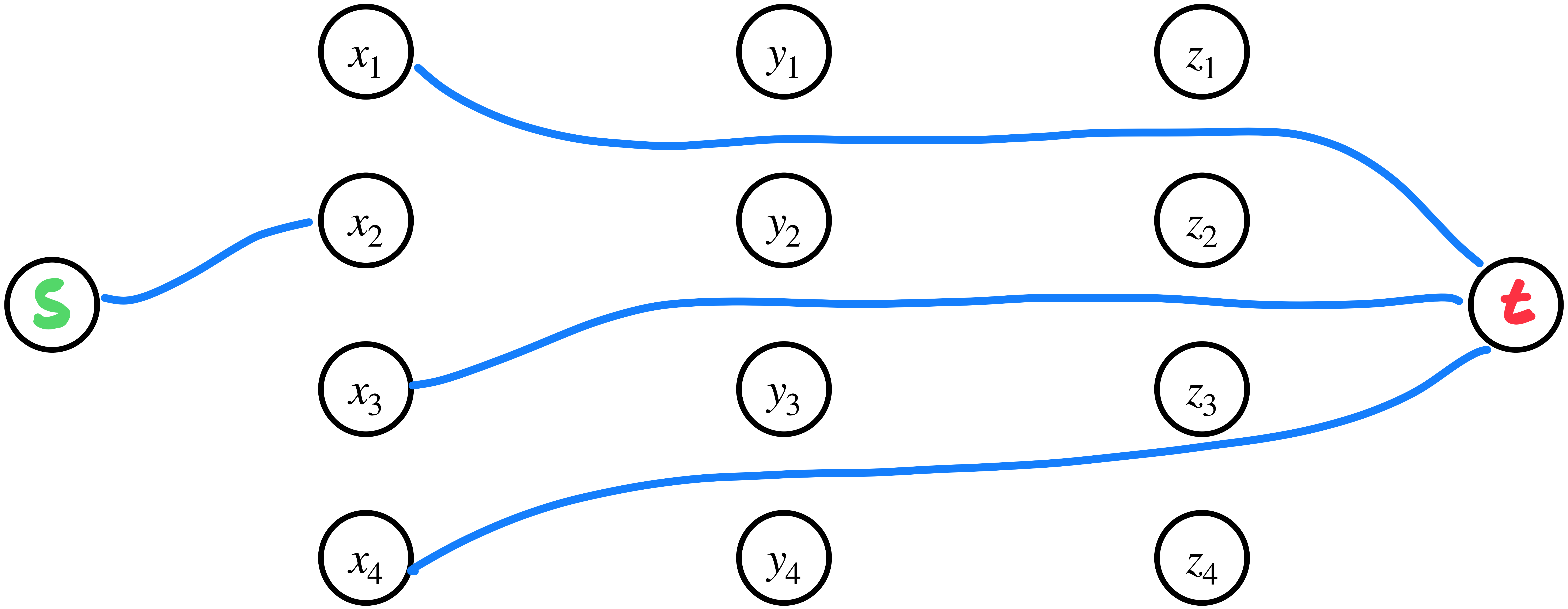

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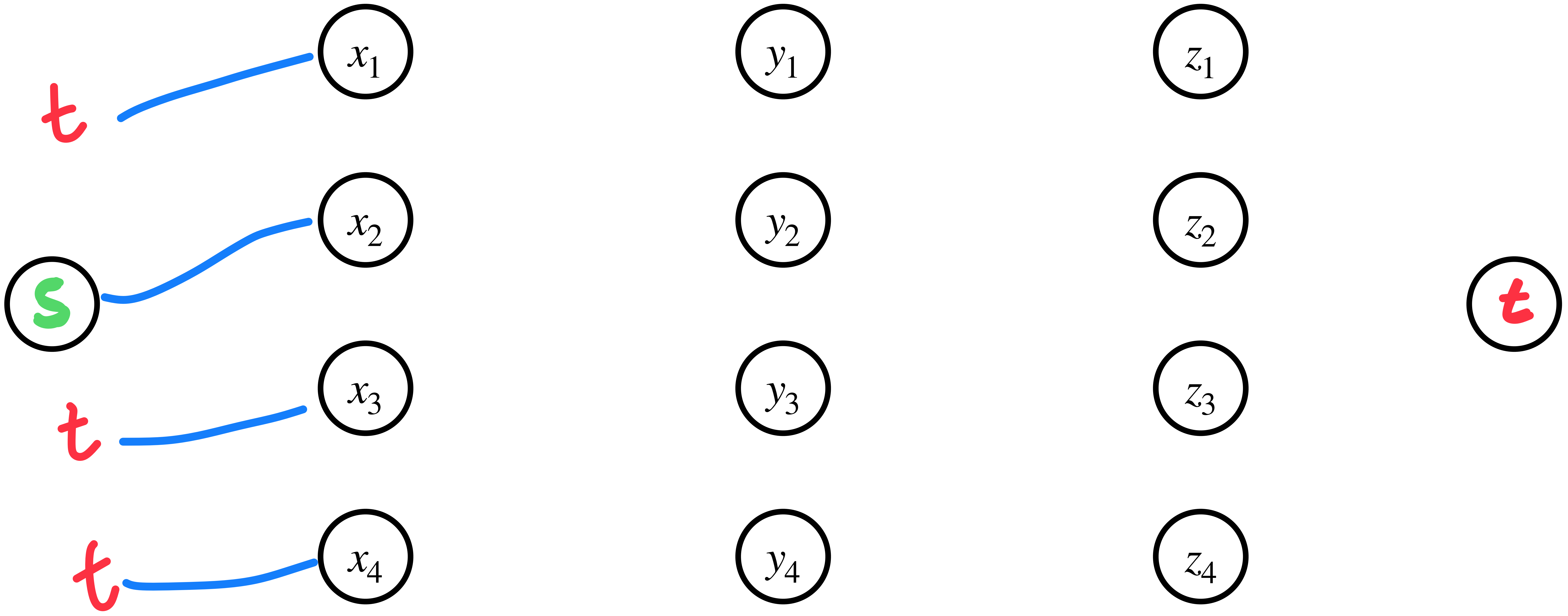

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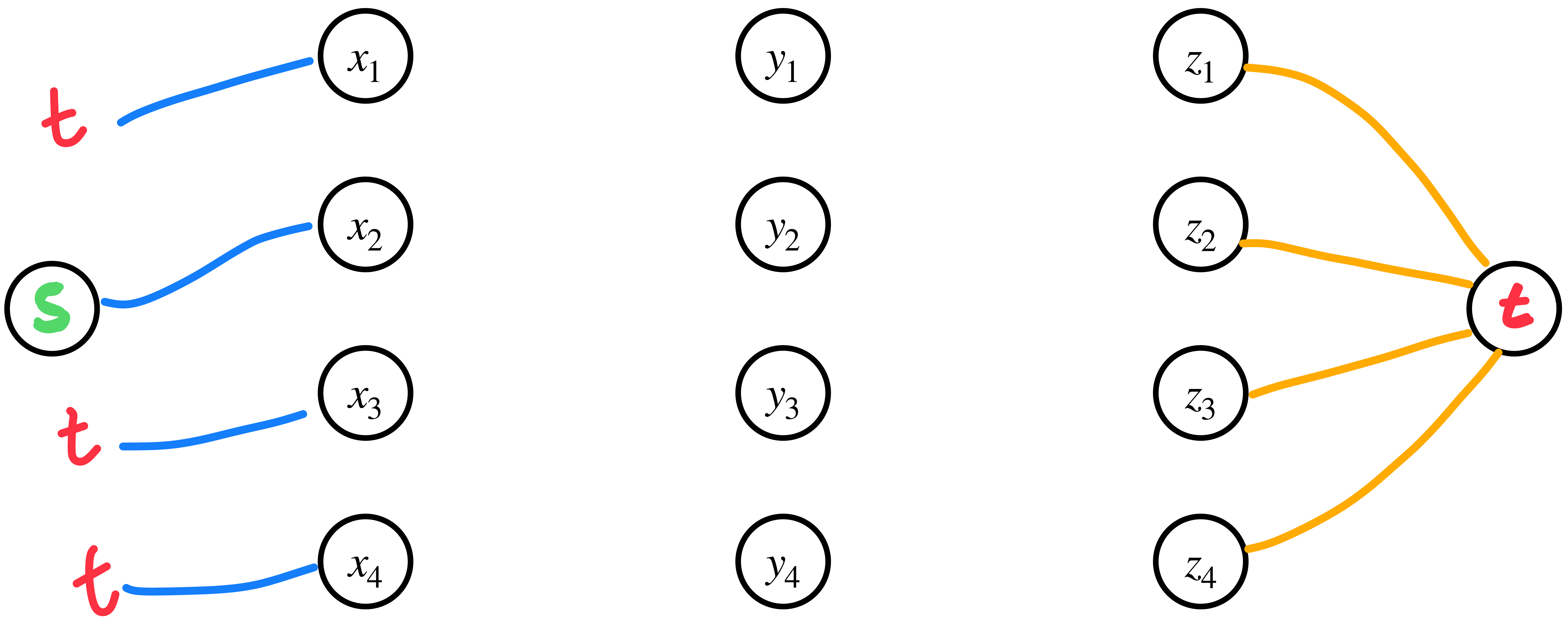

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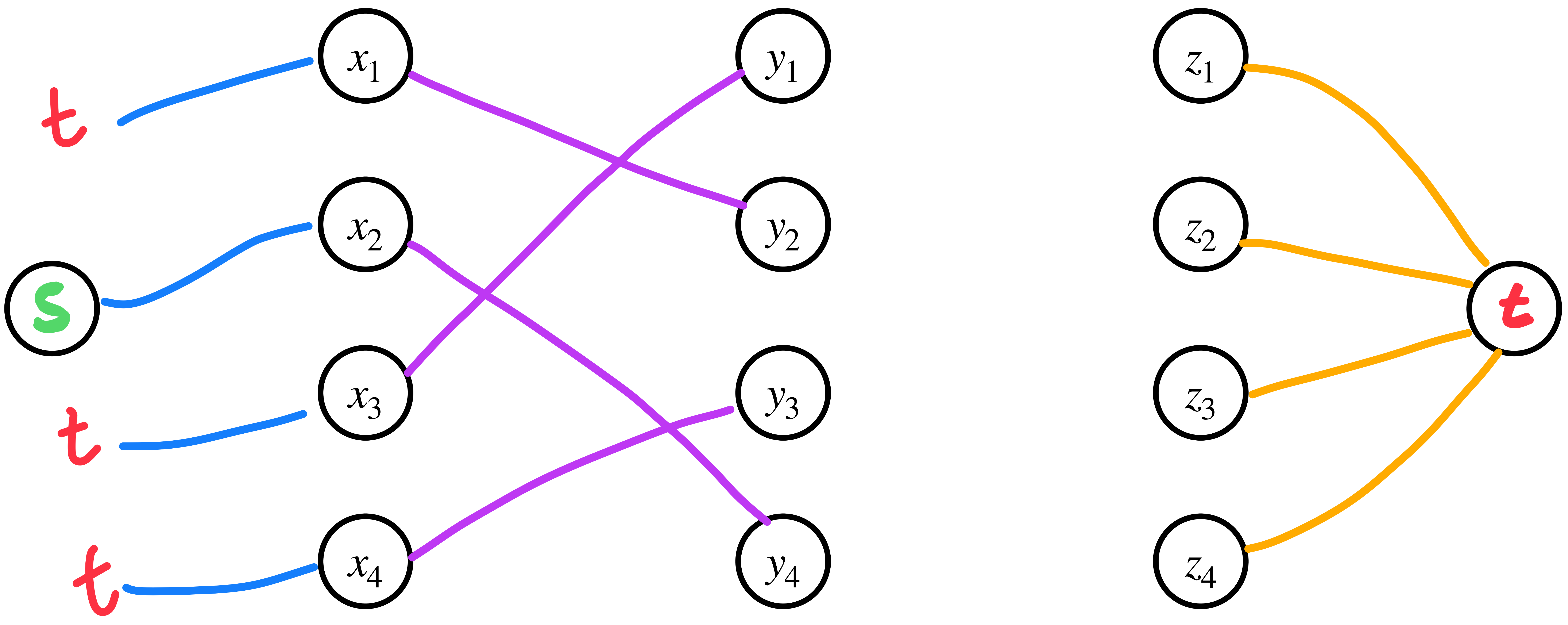

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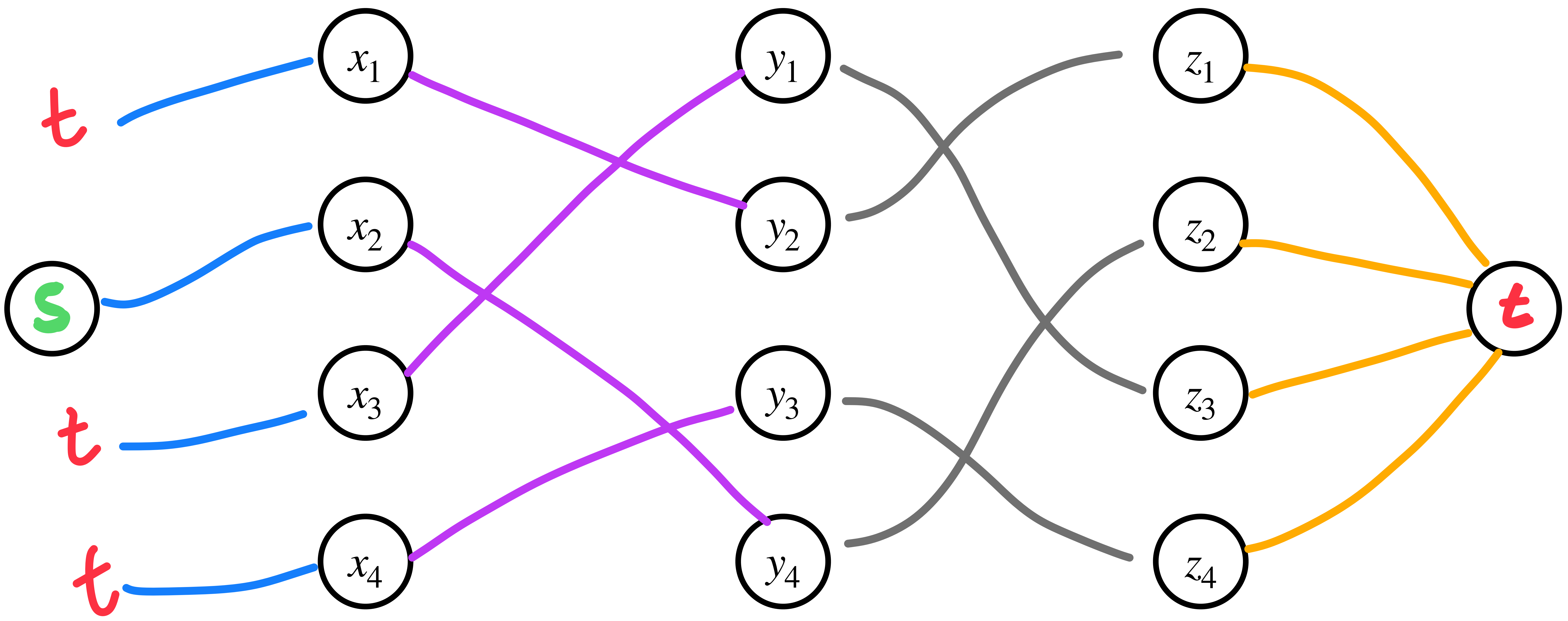

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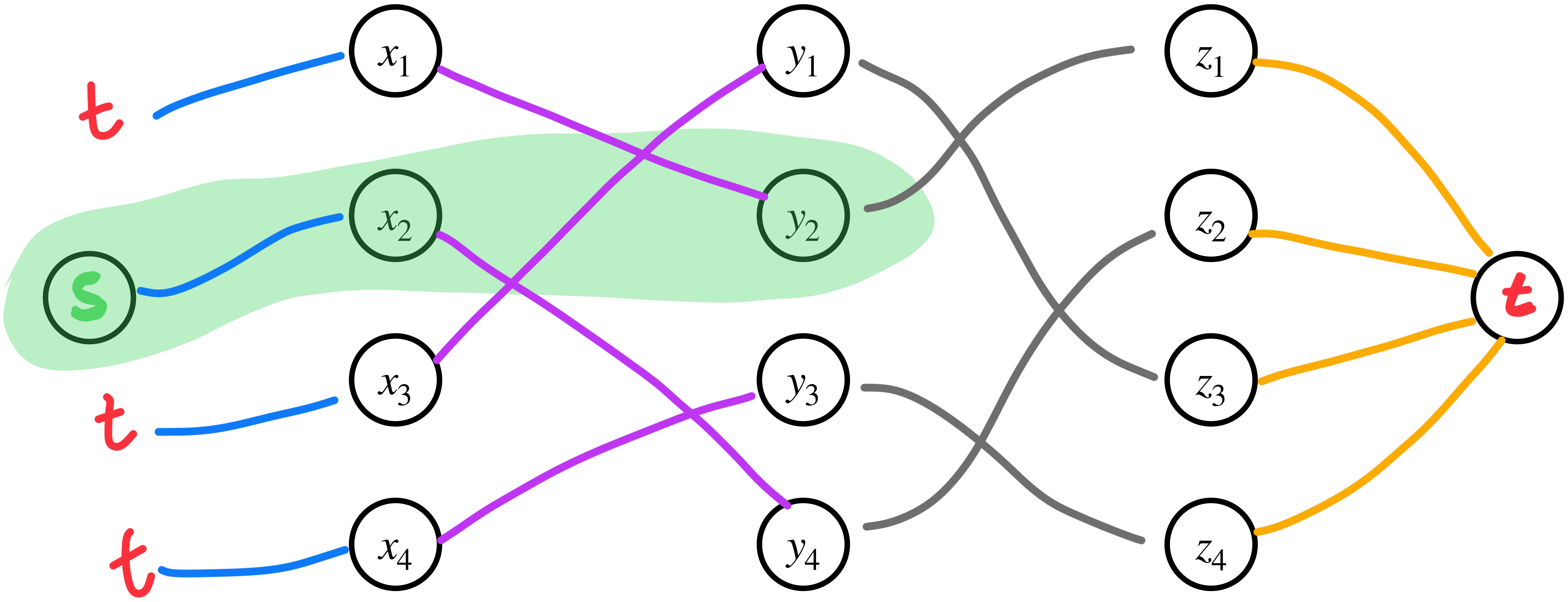
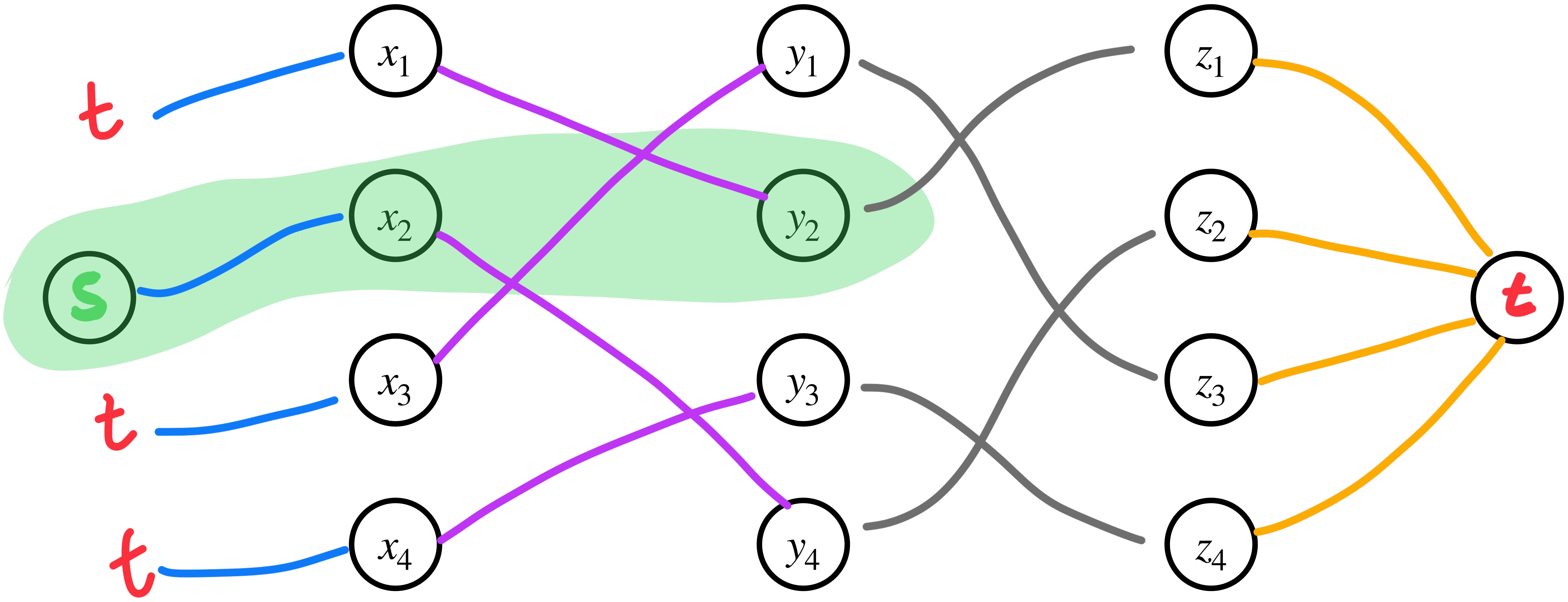


$G = (V, E)$. A st-cut $S \subseteq V$ is **conformal** $\forall v \in X$, at most one $v_a \in S$.

FPT Approximation

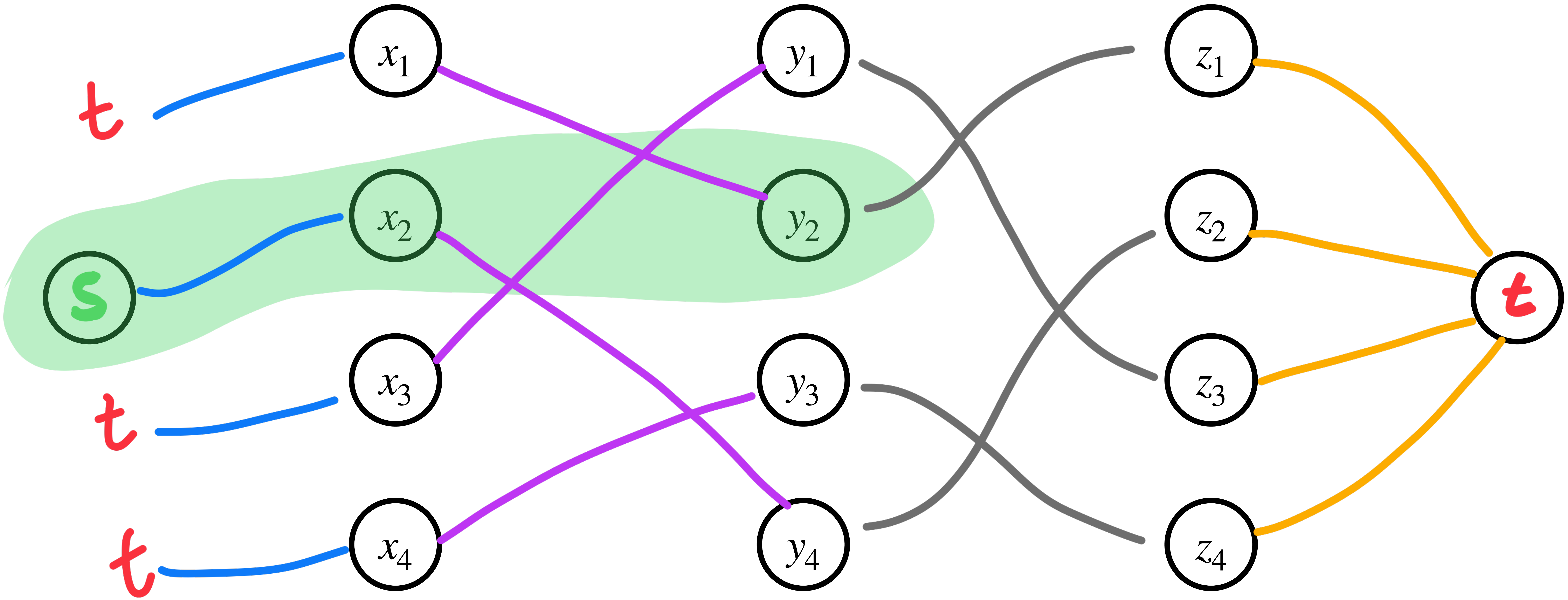
Warm-Up: Algorithm for Fields



Assignments $\alpha : X \to R$ $\overset{\text{1-to-1}}{\longleftrightarrow}$ conformal st-cuts in G

FPT Approximation

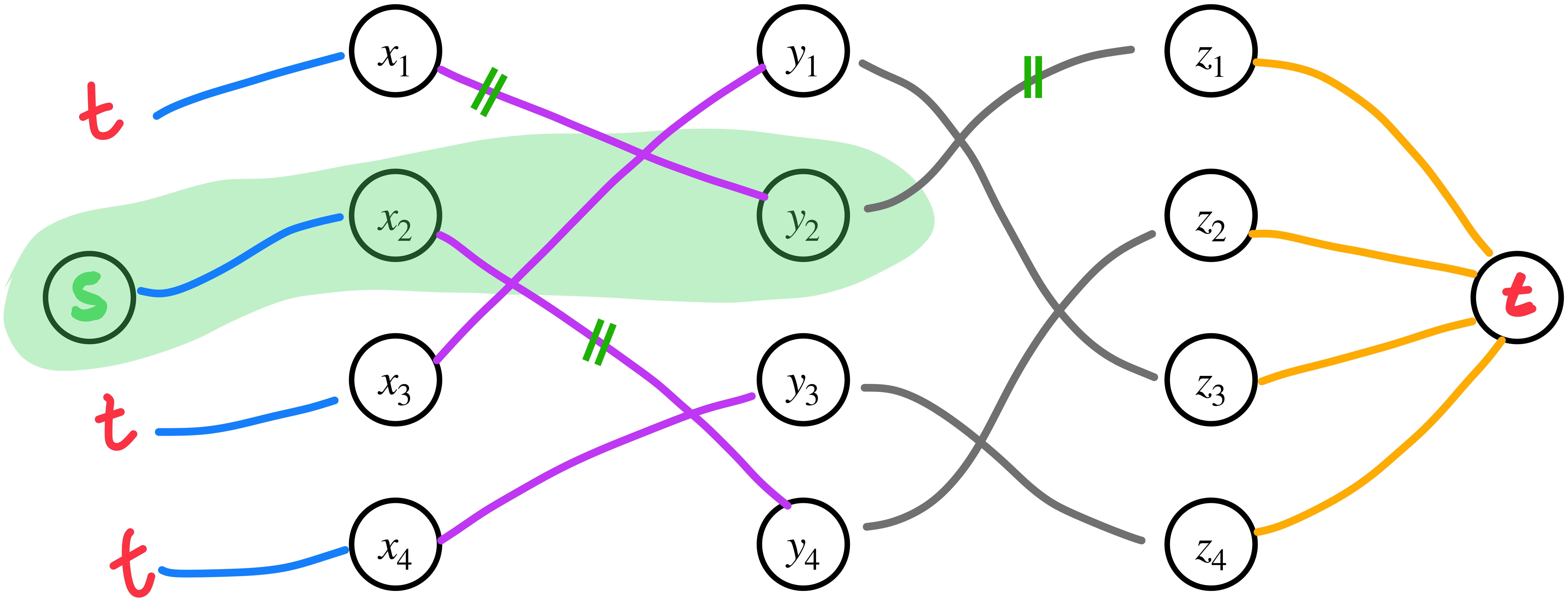
Warm-Up: Algorithm for Fields



$\alpha(x) = 2,\ \alpha(y) = 2,\ \alpha(z) = 0 \quad \longrightarrow \quad$ conformal st-cut $S_\alpha = \{s, x_2, y_2\}$

FPT Approximation

Warm-Up: Algorithm for Fields



Let $\delta(S)$ be the edges cut by S

Key Lemma: $|\delta(S_\alpha)| \leq 2 \cdot \text{cost}(\alpha)$

FPT Approximation for Fields

Warm-Up: Algorithm for Fields

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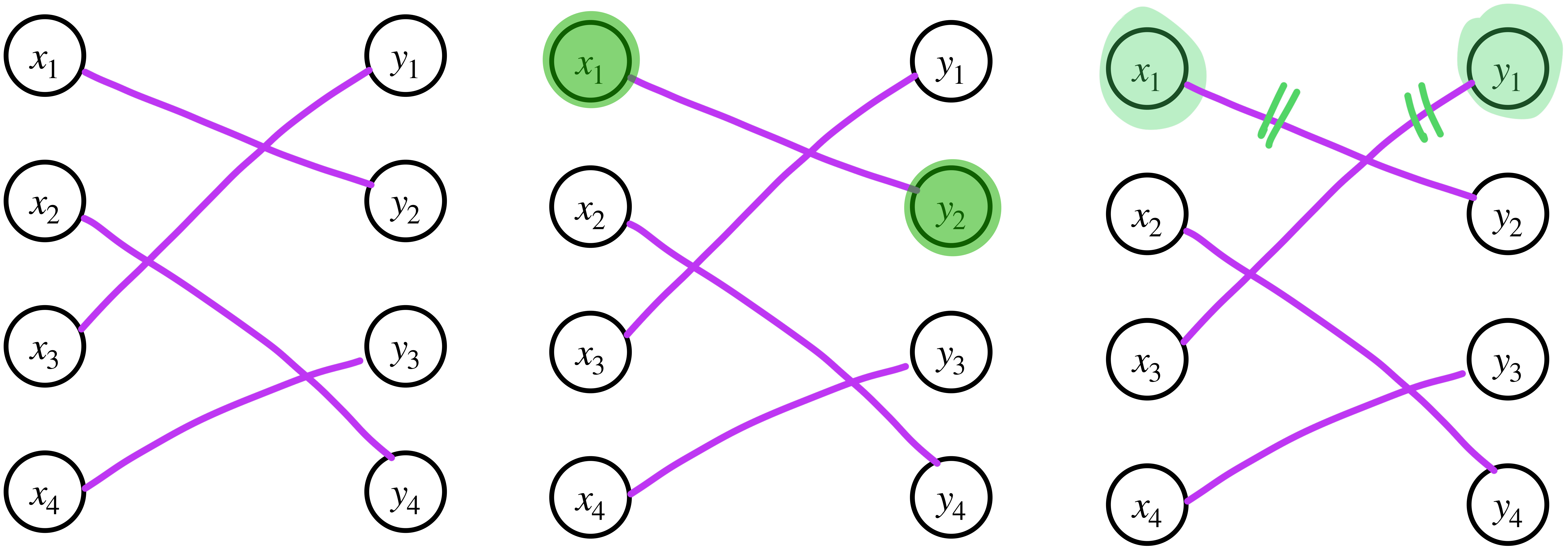


FPT Approximation for Fields

Blueprint

An FPT 2-approximation for Min-2-Lin($\mathbb{Z}_p$):

1) Iterative compression, branching and homogenization …

2) Construct the graph G.

3) Find a conformal st-cut S with $|\delta(S)| \leq 2k$ using *DPC*[1] branching.

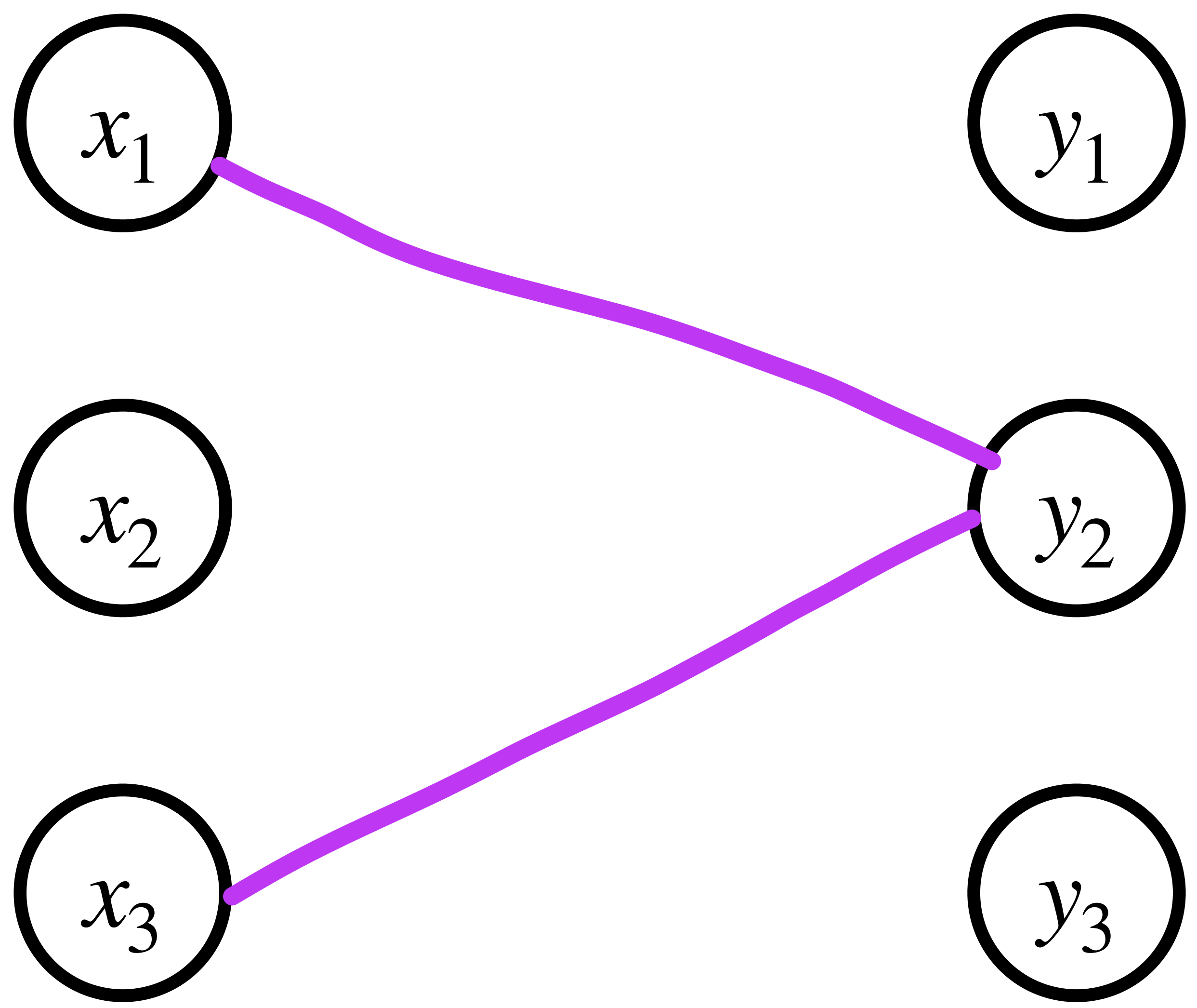
4) Construct an assignment from S and return it.

DPC = Digraph Pair Cut [KW JACM'20]

FPT Approximation for Modular Rings

What goes wrong

Consider an equation $2x = y \mod 4$

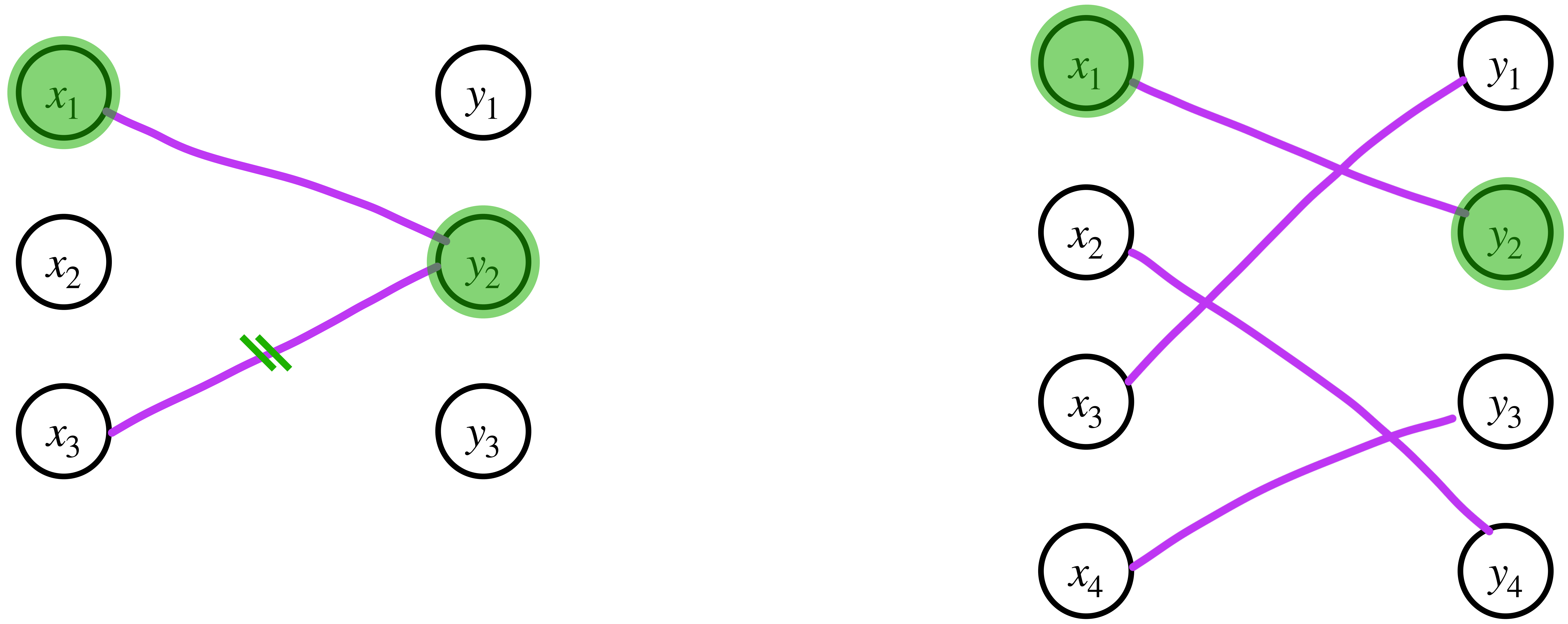


A constraint is no longer a matching…

FPT Approximation for Modular Rings

What goes wrong

$2x = y \mod 4$ vs $2x = 3y \mod 5$



FPT Approximation for Modular Rings

Idea

Create vertices x_C for *sets* of values $C \subseteq \mathbb{Z}_{p^n}$ rather than single values.

Classes are chosen to

1) preserve matching structure of constraints, and

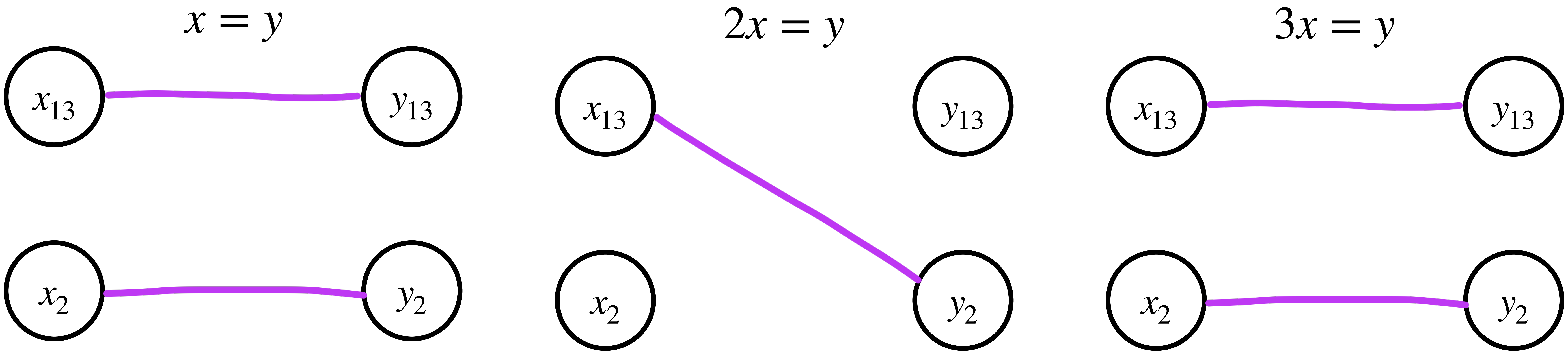
2) allow recursion:

$$\mathbb{Z}_{p^n} \to \mathbb{Z}_{p^{n-1}} \to \mathbb{Z}_{p^{n-2}} \to \ldots \to \mathbb{Z}_p$$

FPT Approximation for Modular Rings

Matching structure

$\mathbb{Z}_4$ classes: $\{0\}$, $\{1,3\}$, $\{2\}$.



Edges form a matching again!

FPT Approximation for Modular Rings

Recursive step

Consider an equation $3x = y \mod 4$.

Our *class* assignment: $x \to \{1,3\}$, $y \to \{1,3\}$.

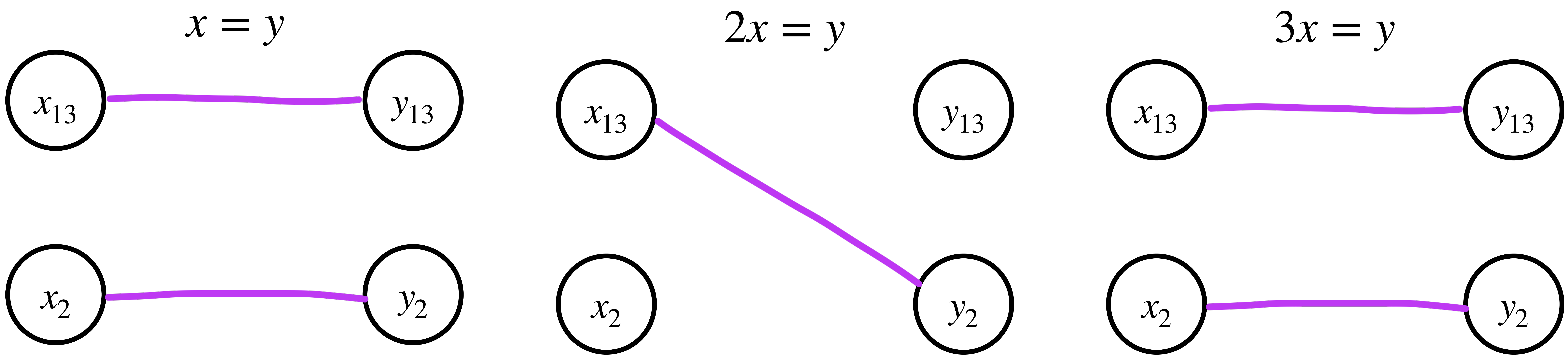
Rewrite into $\mathbb{Z}_2$:

- $x = 2x' + 1,\ y = 2y' + 1$
- $3x = y \text{ mod } 4 \iff 2x' = 2y' + 2 \text{ mod } 4 \iff x' = y' + 1 \text{ mod } 2.$

FPT Approximation for Modular Rings

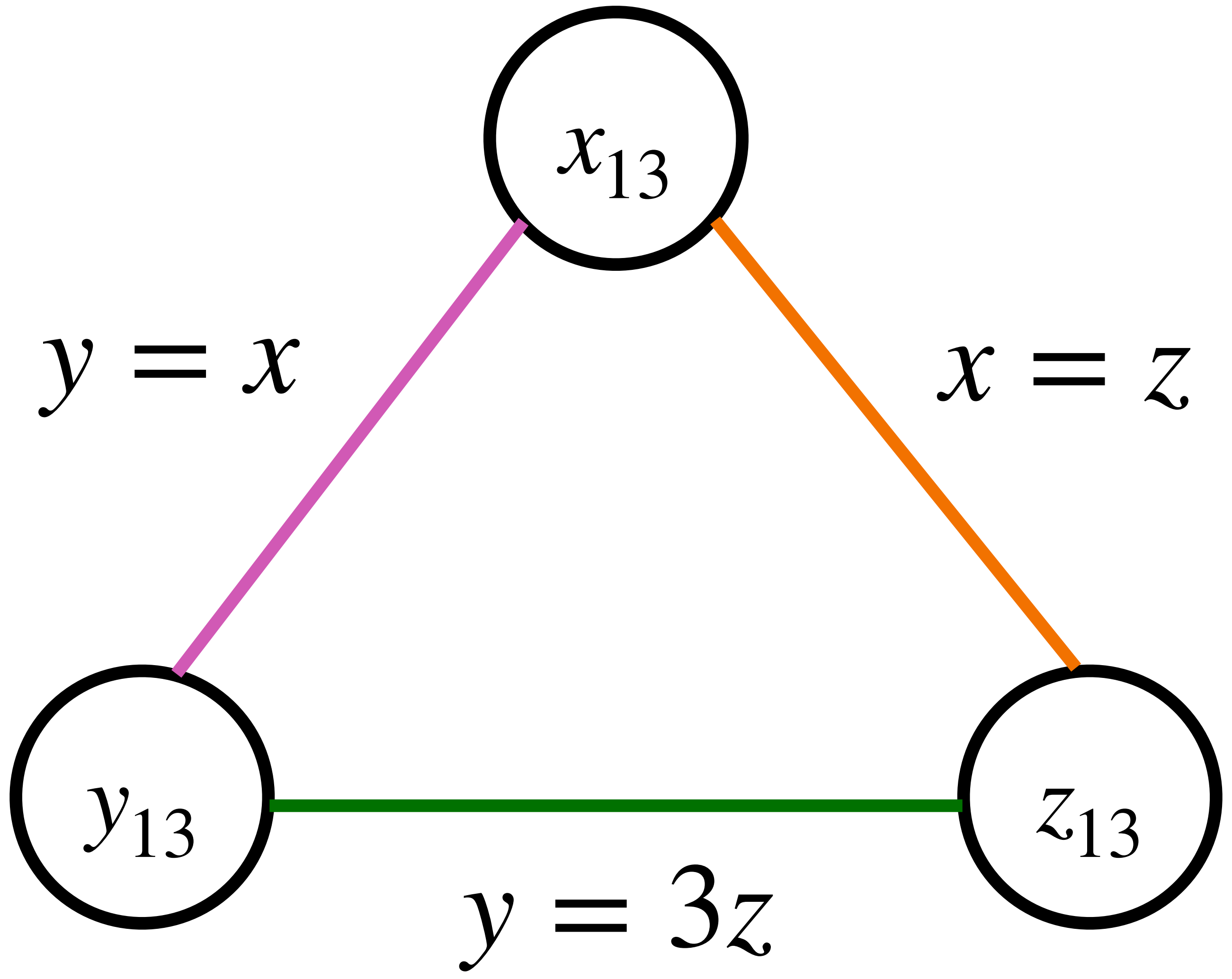
Information lost?

$\mathbb{Z}_4$ classes: $\{0\}, \{1,3\}, \{2\}$.

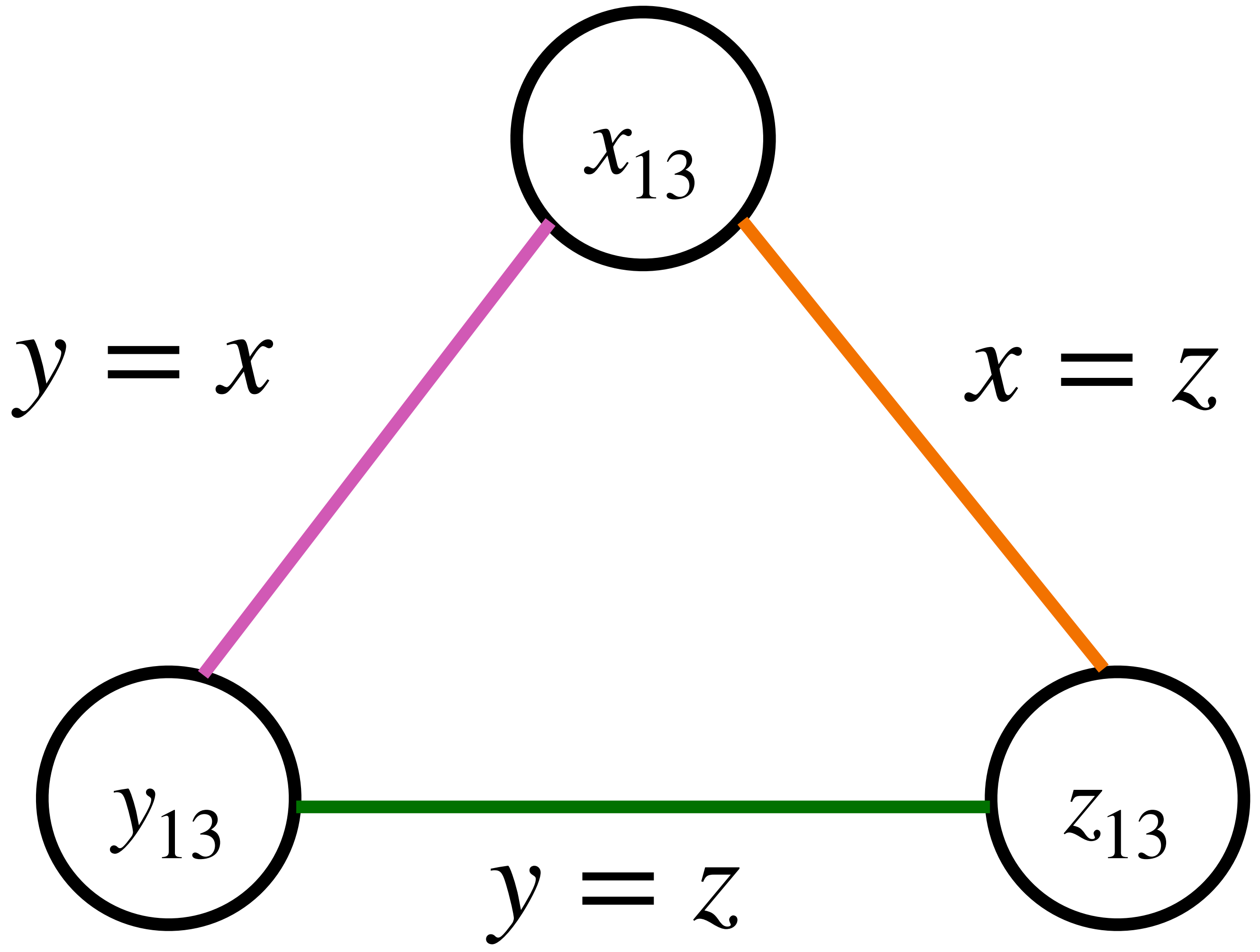


FPT Approximation for Modular Rings

Information lost?

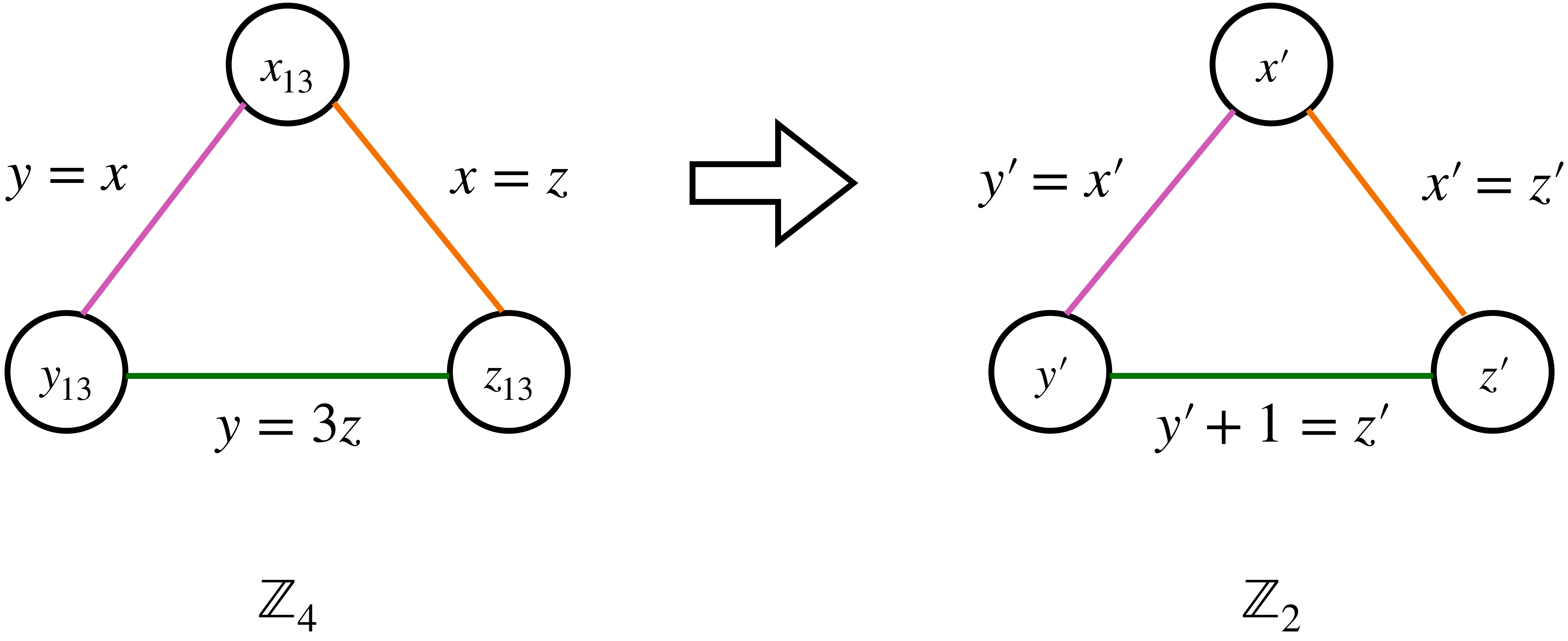


vs



FPT Approximation for Modular Rings

Information preserved



FPT Approximation for Modular Rings

Summary

An FPT **O(1)**-approximation for Min-2-Lin($\mathbb{Z}_{p^n}$):

1) Iterative compression, branching and homogenization…

2) Construct the class assignment graph G.

3) Find a conformal st-cut S with $|\delta(S)| \leq 2k$ **that guarantees progress.** 👈

4) Construct a class assignment from S, rewrite equations into $\mathbb{Z}_{p^{n-1}}$.

* DPC branching fails. Need to use *shadow removal* + complicated branching.

Summary and Open Problems

Theorem: Min-2-Lin($\mathbb{Z}_m$) is $2\omega(m)$-approximable in FPT time.

For example, $2\omega(4) = 2$, so Min-2-Lin($\mathbb{Z}_4$) is 2-approximable in FPT time.

Unpublished: Min-2-Lin($\mathbb{Z}_m$) is W[1]-hard to $(\omega(m) - \varepsilon)$-approximate.

Question 1 (💀): How to close the gap? Is Min-2-Lin($\mathbb{Z}_{p^n}$) in FPT or W[1]-hard?

Question 2 (🌸): Can we remove shadow removal? (👇 LMRSZ SODA'21)

Question 3 (💍): For which finite rings R is Min-2-Lin(R) FPT-apx? (👇 arXiv:2410.09932)

Question 4 (🌊): Which MinCSPs are FPT-approximable?

Thank you!