Resolving Inconsistencies in Disjunctive Temporal Constraints

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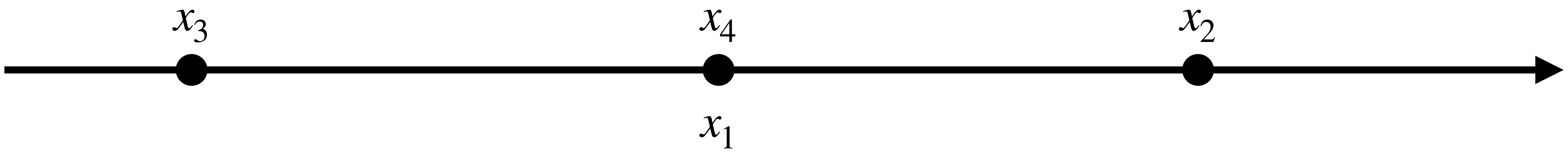
Temporal Reasoning Formalisms

Objects: points on the timeline $x_1, x_2, \ldots, x_n$



Point Algebra: $x_3 < x_1 \quad x_1 = x_4 \quad x_3 \neq x_2 \quad x_3 \leq x_4$

STP: $0 \leq x_1 - x_3 \leq 3 \quad 0 \leq x_1 - x_4 \leq 0 \quad -\infty \leq x_3 - x_2 \leq 10$

2-DTP: $0 \leq x_1 - x_3 \leq 1 \vee 3 \leq x_1 - x_3 \leq 4$

Refs: Vilain, Kautz [AAAI'86], Dechter, Meiri, Pearl [KR'89]

Temporal Reasoning Formalisms

More Formally

Point Algebra (PA): contains binary relations $<, >, \leq, \geq, \neq, =$ over $\mathbb{Q}$.

STP: contains all binary relations $x - y \in [a, b]$ where $a, b \in \mathbb{Q} \cup \{-\infty, +\infty\}$.

2-DTP: contains all binary relations $x - y \in [a_1, b_1] \cup \ldots \cup [a_\ell, b_\ell]$.

Checking Consistency

CSP(Γ) // **c**onstraint **s**atisfaction **p**roblem

In: variables $X = (x_1, \ldots, x_n)$,
constraints of the form $x_i \ R \ x_j$ for some $x_i, x_j \in X$ and $R \in \Gamma$

Out: is there an assignment $\alpha : X \to \mathbb{Q}$ that satisfies all constraints?

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Example 1: CSP(*PA*) instance $x_3 < x_1, \quad x_1 = x_4, \quad x_2 \neq x_3, \quad x_3 \leq x_4$

Assignment $\alpha(x_1) = 2$, $\alpha(x_2) = 3$, $\alpha(x_3) = 1$, $\alpha(x_4) = 2$ satisfies all constraints.

Checking Consistency

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Out: is there an assignment $\alpha : X \to \mathbb{Q}$ that satisfies all constraints?

Example 2: CSP(*STP*) instance

$$1 \leq x_1 - x_2 \leq 2$$
$$2 \leq x_2 - x_3 \leq 3$$
$$-\infty \leq x_1 - x_3 \leq 2$$

Unsatisfiable

Fixing a Few Errors

MinCSP(Γ)

In: instance (X, C) of CSP(Γ) and budget $k \in \mathbb{N}$.

Out: is there $F \subseteq C$, $|F| \leq k$ such that $(X, C \backslash F)$ is satisfiable?

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Example 3: CSP(*STP*) instance $1 \leq x_1 - x_2 \leq 2$, $2 \leq x_2 - x_3 \leq 3$, $-\infty \leq x_1 - x_3 \leq 2$

No solution for $k = 0$.

Fixing a Few Errors

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Example 3: CSP(*STP*) instance $1 \leq x_1 - x_2 \leq 2$, $2 \leq x_2 - x_3 \leq 3$, $\cancel{-\infty \leq x_1 - x_3 \leq 2}$

No solution for $k = 0$.

Solution for $k = 1$: $F = \{-\infty \leq x_1 - x_3 \leq 2\}$

Computational Complexity

	CSP	**MinCSP**
Point Algebra	in P	**NP-hard**
STP	in P	**NP-hard**
2-DTP	NP-hard	**NP-hard**

Fine-Grained Analysis

Restricting Expressive Power

$\Gamma \in \{$*PA, STP, 2-DTP*$\}$

Question: What is the complexity of MinCSP(Γ') for all $\Gamma' \subseteq \Gamma$?

Fine-Grained Analysis

Restricting Expressive Power

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Question: What is the complexity of MinCSP(Γ') for all $\Gamma' \subseteq \Gamma$?

Answer: MinCSP(Γ') remains NP-hard except for “trivial” cases.

🙁

Parameterized Complexity

MinCSP(Γ)

In: instance (X, C) of CSP(Γ) and budget $k \in \mathbb{N}$.

Out: is there $F \subseteq C$, $|F| \leq k$ such that $(X, C \backslash F)$ is satisfiable?

Can we at least solve MinCSP(Γ) when k is small?

Parameterized Complexity

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Can we at least solve MinCSP(Γ) when k is small?

pNP-h: CSP(Γ) is NP-hard (hard even for $k = 0$)

XP: if CSP(Γ) is in P, try all k-subsets F. Requires $|C|^{O(k)}$ time.

FPT: can we solve it in $f(k) \cdot |C|^{O(1)}$ time? // $f(k)$ could be 2^k, 3^k, $k!$, 2^{2^k}

W[1]-h: as hard as finding a k-Clique (strong evidence $\notin$ FPT)

Parameterized Complexity Classification

Restricting Expressive Power

$\Gamma \in \{PA, STP, 2\text{-}DTP\}$

Question: What is the parameterized complexity of MinCSP(Γ') for all $\Gamma' \subseteq \Gamma$?

- [OPW ESA'23] classified MinCSP(Γ') for all $\Gamma' \subseteq$ *PA*.
- [DJOO AAAI'22] classified MinCSP(Γ') for all $\Gamma' \subseteq$ *STP*.
- This work: classified MinCSP(Γ') for all $\Gamma' \subseteq$ *2-DTP*.

Classification for 2-DTP

Recall: *2-DTP* relations $x - y \in [a_1, b_1] \cup \ldots \cup [a_\ell, b_\ell]$ where $a_i, b_i \in \mathbb{Q} \cup \{-\infty, +\infty\}$

Question: What is the parameterized complexity of MinCSP(Γ') for all $\Gamma' \subseteq$ *2-DTP*?

Classification for 2-DTP

WLOG

Recall: *2-DTP* relations $x - y \in [a_1, b_1] \cup \ldots \cup [a_\ell, b_\ell]$ where $a_i, b_i \in \mathbb{Z} \cup \{-\infty, +\infty\}$

Question: What is the parameterized complexity of MinCSP(Γ') for all $\Gamma' \subseteq$ *2-DTP*?

Classification Theorem

Based on [BMM JACM'18]

2-DTP relations $x - y \in [a_1, b_1] \cup \ldots \cup [a_\ell, b_\ell]$ where $a_i, b_i \in \mathbb{Z} \cup \{-\infty, +\infty\}$

For every $\Gamma' \subseteq$ *2-DTP*, there exists B such that CSP(Γ') = CSP(B) and OOTFH:

1. B has a finite domain
2. B ⊆ *PA*
3. B ⊆ *STP*
4. B is a set of periodic 2-DTP relations
5. B is a set of equations and disequations
6. CSP(B) is NP-hard

Classification Theorem

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Classification Theorem

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Finite Domain Case Example

$$|x_1 - x_2| = 1 \quad \approx \quad x_1 - x_2 \in [-1, -1] \cup [1,1]$$

$$|x_2 - x_3| = 1$$

$$|x_3 - x_4| = 1$$

Claim: satisfiable over $\mathbb{Z}$ iff satisfiable over $\{0,1\}$

$$\alpha(x_1) = 1 \;\; \alpha(x_2) = 2 \;\; \alpha(x_3) = 3 \;\; \alpha(x_4) = 4$$

$$\beta(x_1) = 1 \;\; \beta(x_2) = 0 \;\; \beta(x_3) = 1 \;\; \beta(x_4) = 0$$

Classification Theorem

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Finite Domain Case Classification?

B has finite domain.

CSP(**B**) dichotomy: Bulatov [FOCS'17], Zhuk [JACM'20]

MinCSP(**B**) dichotomy:
- P vs NP-h: Thapper & Zivny [JACM'16]
- FPT vs W[1]-h: wide open

Classification Theorem

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Point Algebra Case
Example

$x_1 - x_2 \in [1,\infty]$	$x_1 > x_2$
$x_2 - x_3 \in [0,\infty]$	$x_2 \geq x_3$
$x_3 - x_4 \in [-\infty, -1] \cup [1,\infty]$	$x_3 \neq x_4$

Classification Theorem

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Point Algebra Case
Classification

Let **B** $\subseteq$ ***PA*.**

CSP(**B**) is in P for all **B**.

MinCSP(**B**) is either trivial or NP-hard.

MinCSP(**B**) is FPT unless it contains both $\leq$ and $\neq$, otherwise W[1]-hard

[OPW ESA'23]

Classification Theorem

2-DTP relations $x - y \in [a_1, b_1] \cup \ldots \cup [a_\ell, b_\ell]$ where $a_i, b_i \in \mathbb{Z} \cup \{-\infty, +\infty\}$

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Classification Theorem

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STP Case

STP is *2-DTP* without disjunctions

CSP($\mathbf{B}$) is in P.

MinCSP($\mathbf{B}$) is either trivial or NP-hard.

MinCSP($\mathbf{B}$) is mostly W[1]-hard except for two cases:

- all relations are of the form $x - y \in [a, a]$
- all relations are of the form $x - y \in [a, \infty]$ for $a \geq 0$

[DJOO AAAI'22]

Classification Theorem

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Periodic Case Example

$x_1 - x_2 \in \{0,2,4\}$ $\quad$ $x_1 - x_2 \in [0,0] \cup [2,2] \cup [4,4]$

$x_1 - x_3 \in \{6,8\}$ $\quad$ $x_1 - x_3 \in [6,6] \cup [8,8]$

Divide all values by 2

$y_1 - y_2 \in [0,2]$

$y_1 - y_3 \in [3,4]$

Classification Theorem

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Periodic Case Classification

$x_1 - x_2 \in \{0,2,4\}$ $\quad$ $x_1 - x_2 \in [0,0] \cup [2,2] \cup [4,4]$

$x_1 - x_3 \in \{6,8\}$ $\quad$ $x_1 - x_3 \in [6,6] \cup [8,8]$

$y_1 - y_2 \in [0,2]$

$y_1 - y_3 \in [3,4]$

Reduces to STP with finite intervals.

Classification Theorem

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Classification Theorem

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Equations and Disequations Example

$$x_1 - x_2 = 1 \qquad\qquad x_1 - x_2 \in [1,1]$$
$$x_1 - x_3 \neq 2 \qquad\qquad x_1 - x_3 \in [-\infty,1] \cup [3,\infty]$$

CSP(**B**) is in P.
MinCSP(**B**) is either trivial or NP-hard.
MinCSP(**B**) is FPT!

Classification Theorem

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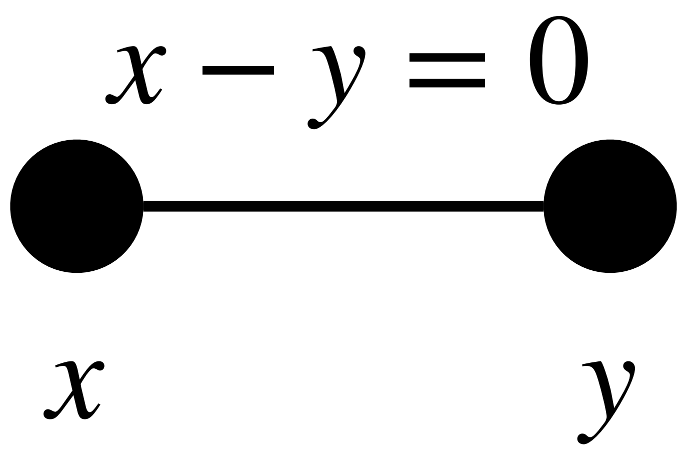
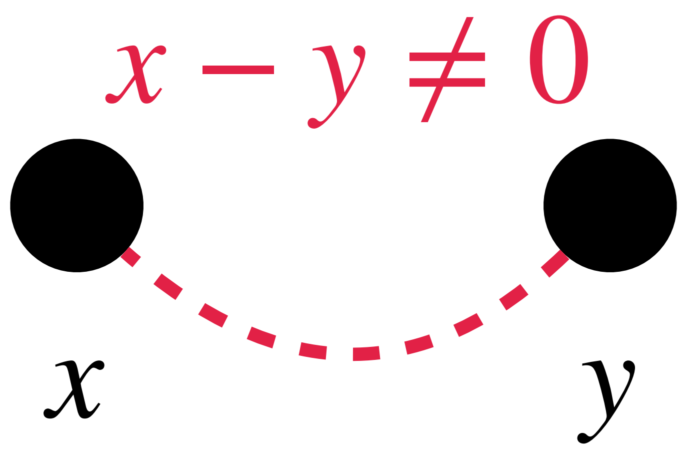
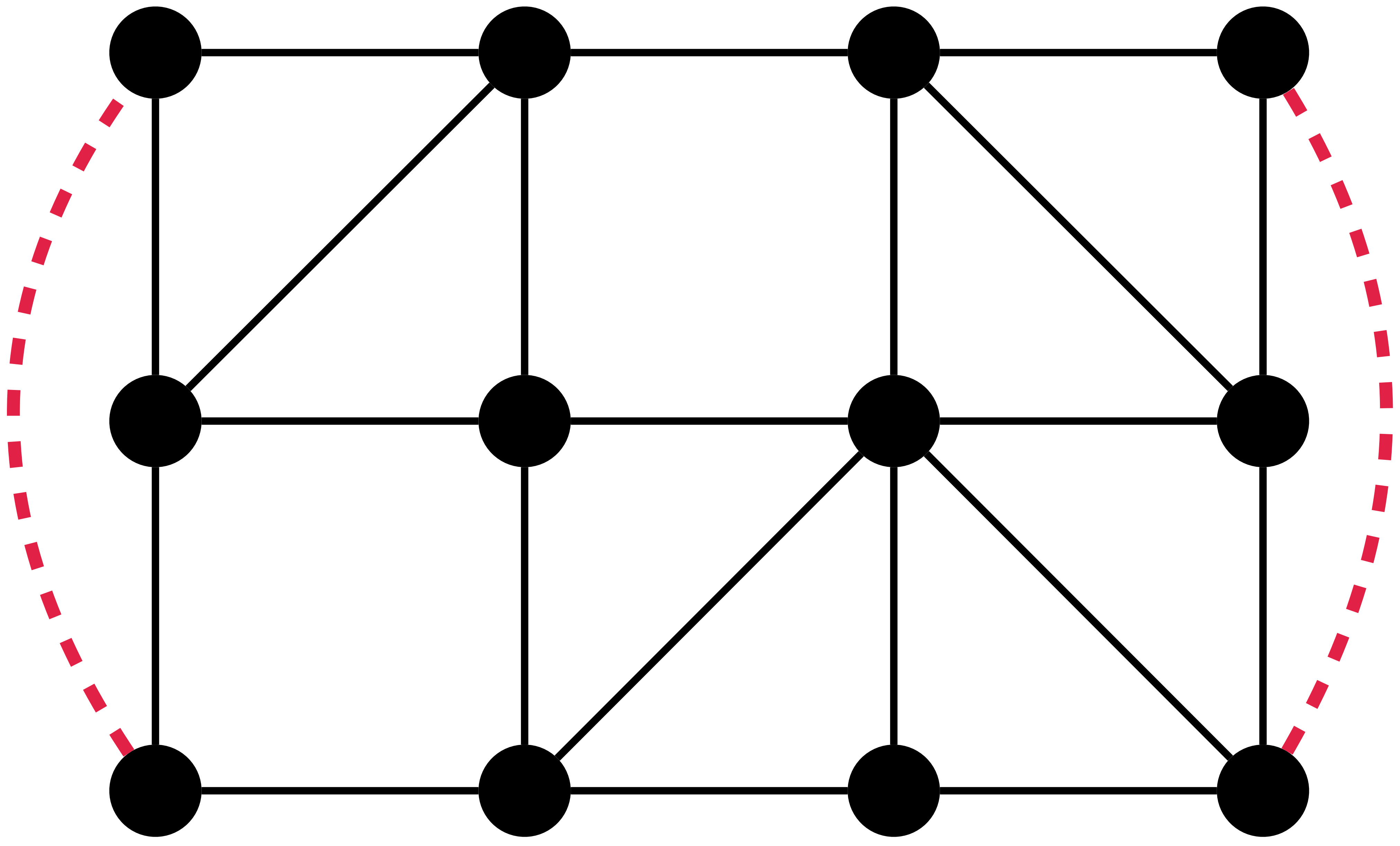
Equations and Disequations

Equations and Disequations

Obstructions of the first kind





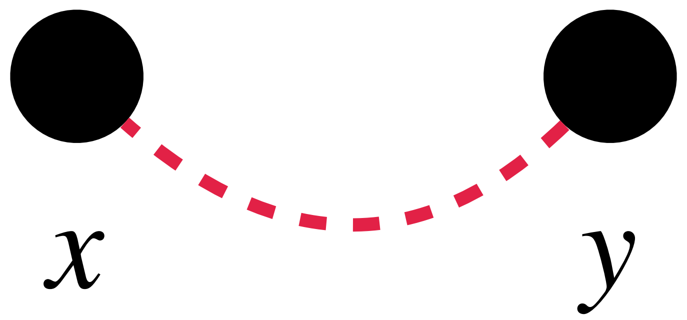


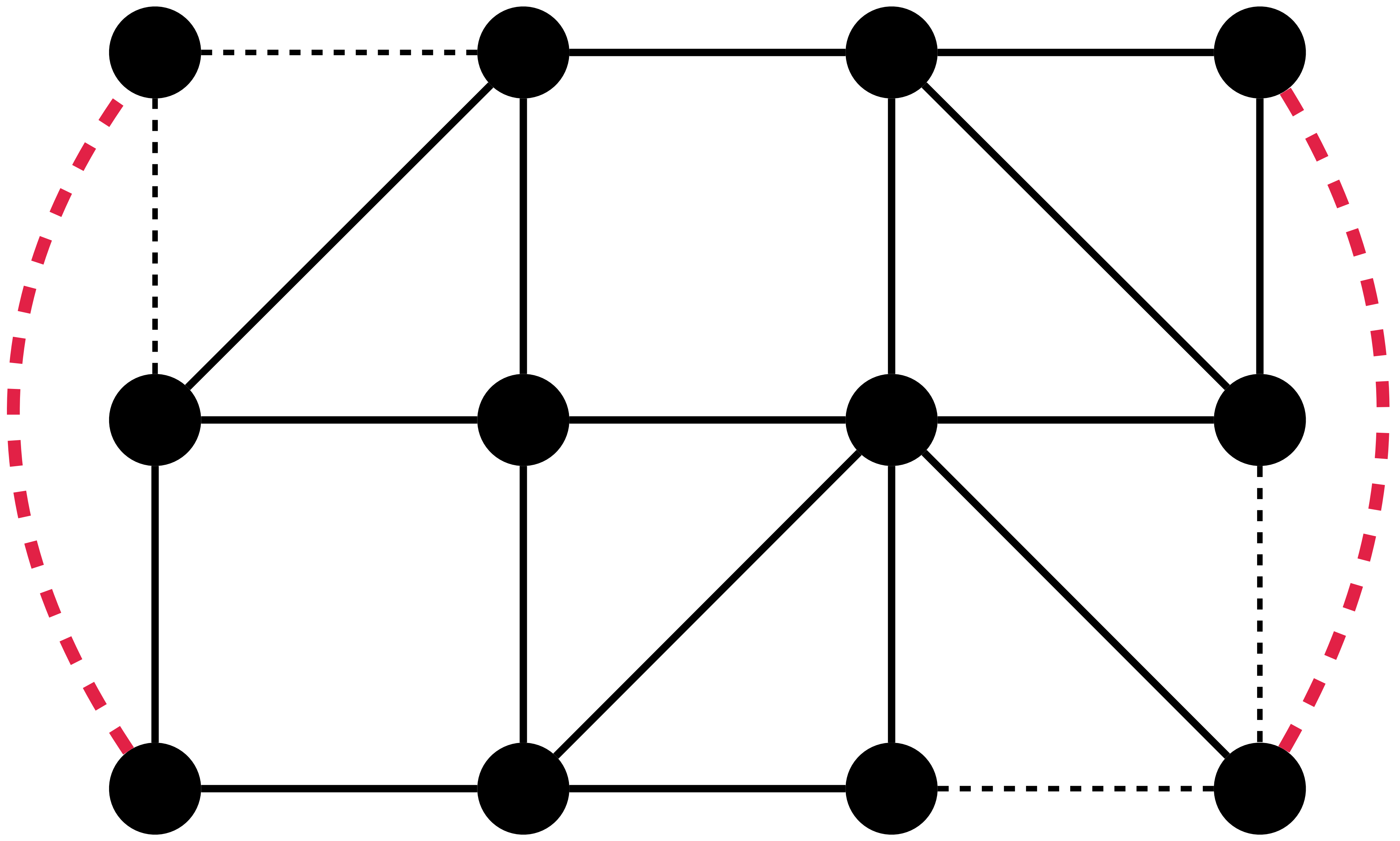
Equations and Disequations

Obstructions of the first kind

$x - y = 0$

$x - y \neq 0$





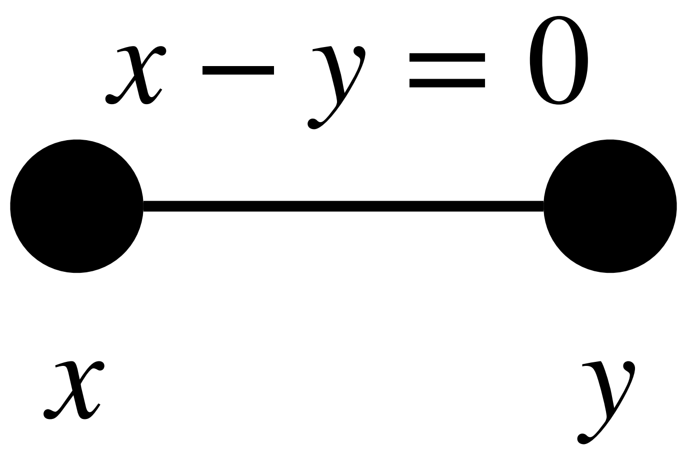
Edge Multicut

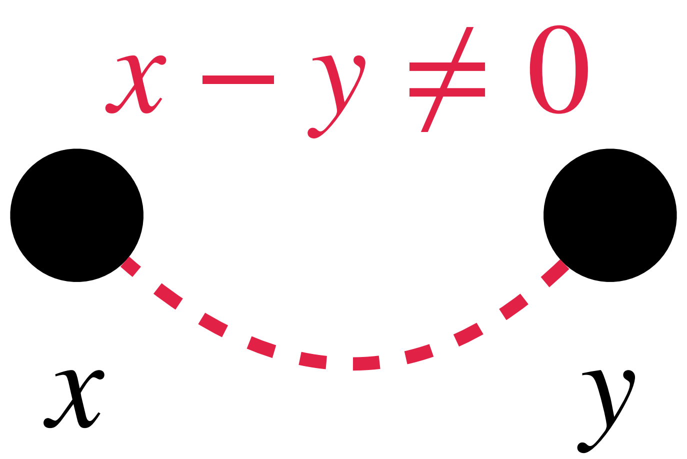
In: graph G, cut requests $(s_1, t_1), \ldots, (s_m, t_m)$

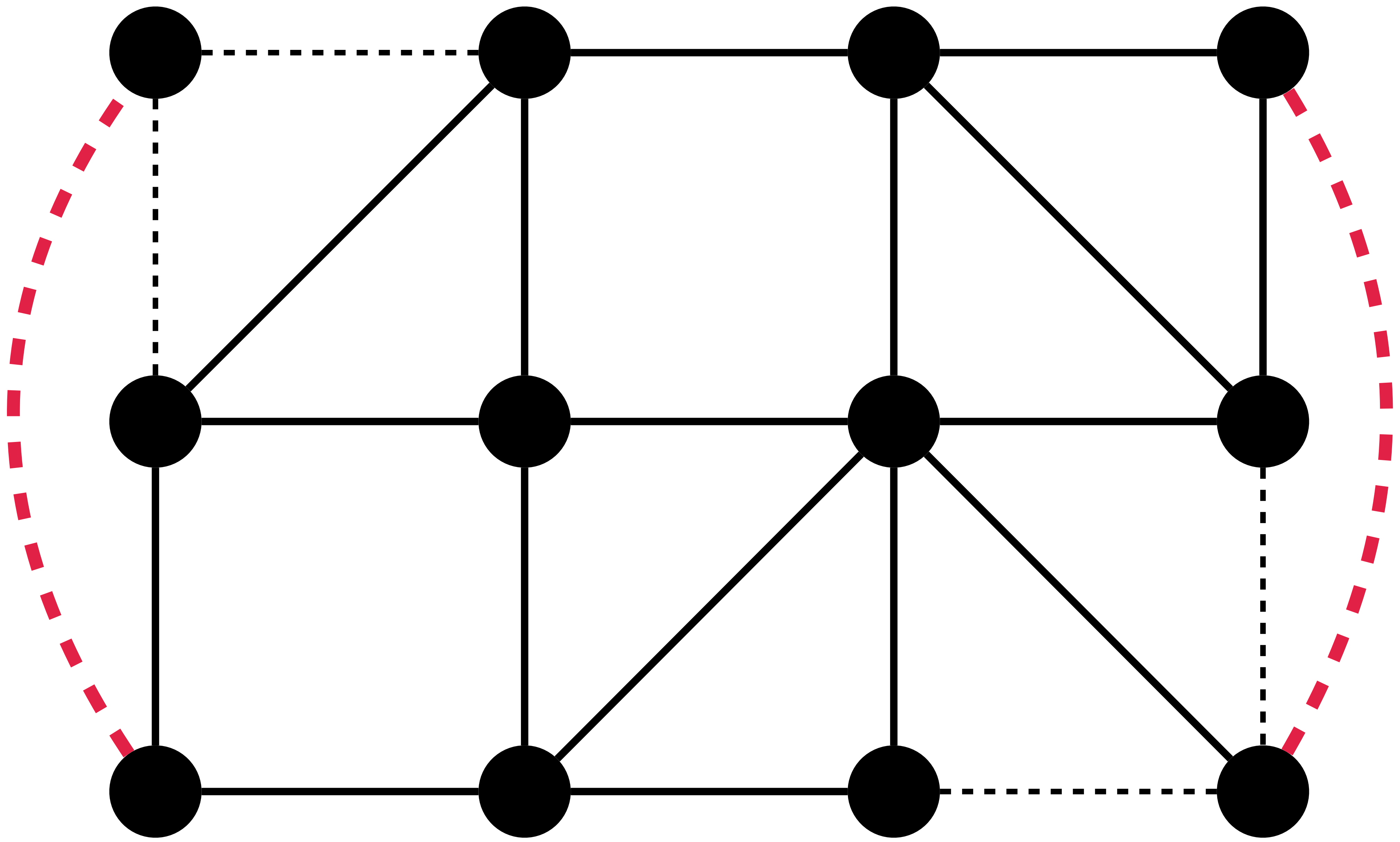
Out: edge subset X of size $\leq k$ that separates all (s_i, t_i)-pairs

Equations and Disequations

Obstructions of the first kind







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FPT! [MR SICOMP'14]

Equations and Disequations

Obstructions of the second kind

$$x_1 - x_2 = 1 \;\equiv\; x_2 + 1 = x_1$$

$$x_2 - x_3 = 1 \;\equiv\; x_3 + 1 = x_2$$

$$x_3 - x_4 = 1 \;\equiv\; x_4 + 1 = x_3$$

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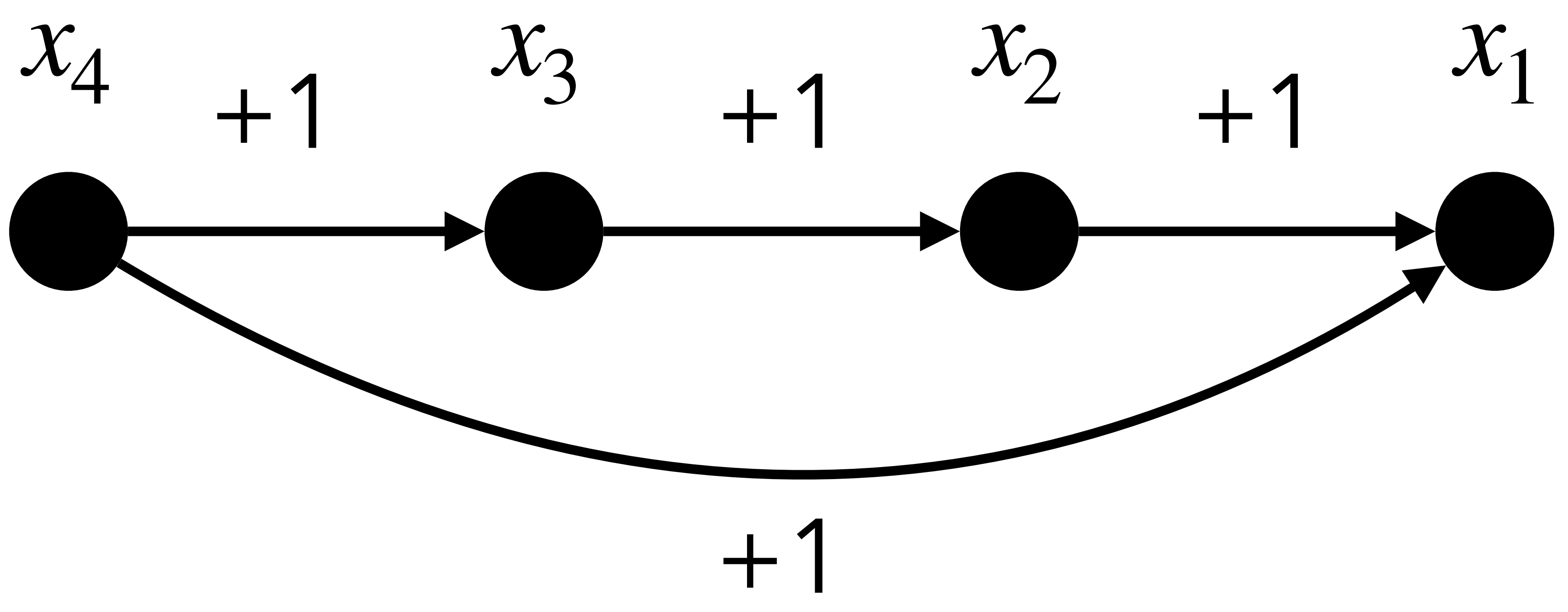

Equations and Disequations

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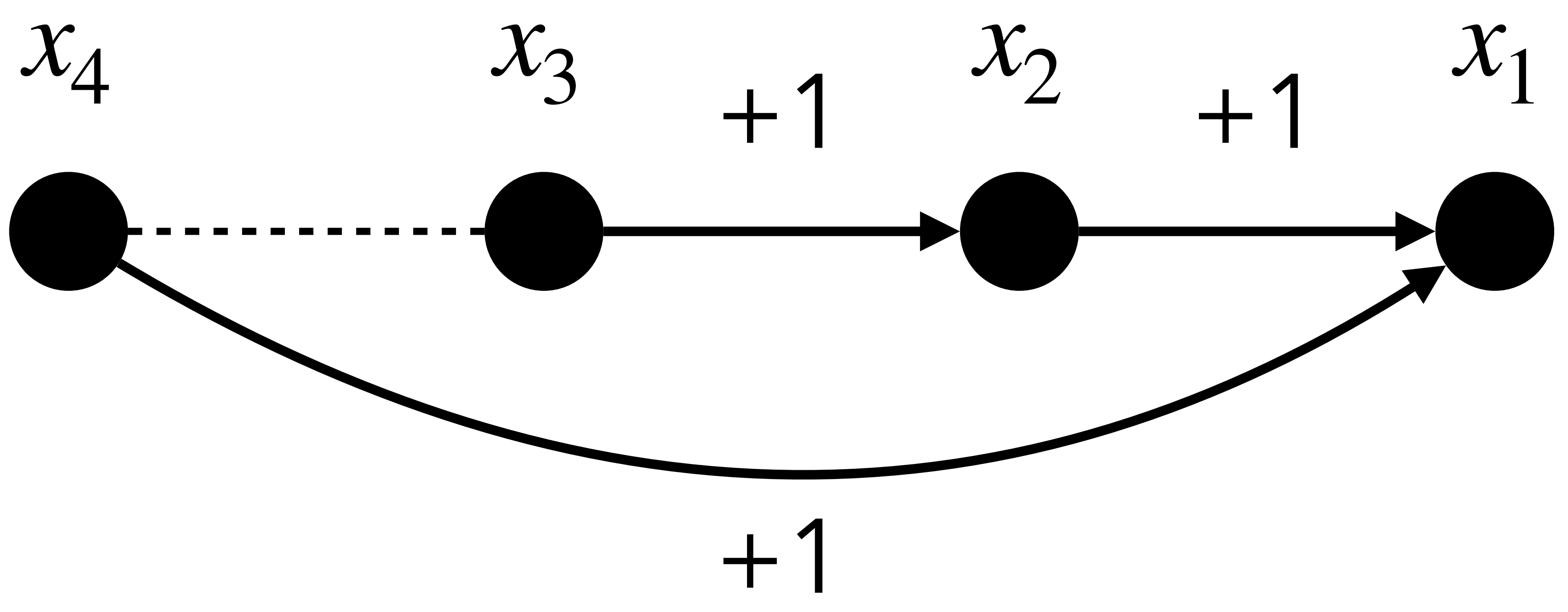

Group Feedback Set

In: a system of difference equations of the form $x - y = 1$
Out: drop at most k equations to make the system satisfiable

Equations and Disequations

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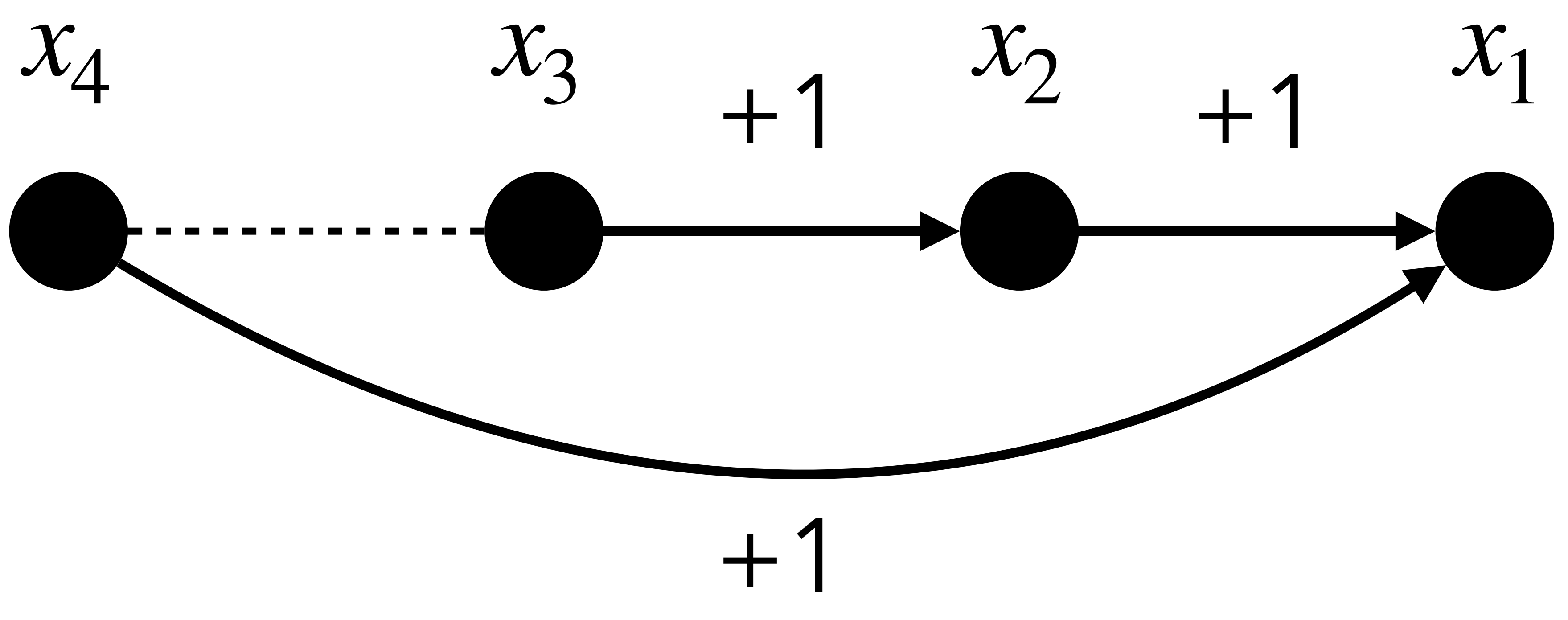


Group Feedback Set

In: a system of difference equations of the form $x - y = 1$

Out: drop at most k equations to make the system satisfiable

FPT!

[CPP Algorithmica'16]

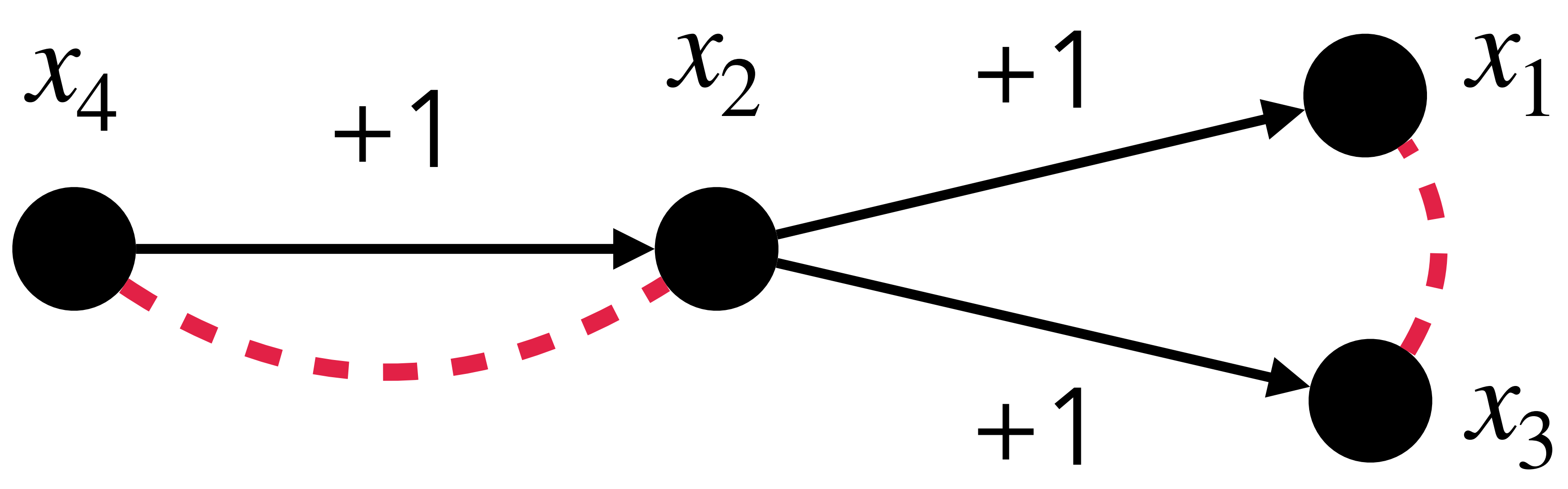
Equations and Disequations

Obstructions together

A mix of difference equations $x = y + 1$ and disequalities $x \neq y$.

One algorithm to handle both types of obstructions.

$$
\begin{aligned}
x_1 - x_2 &= 1 \\
x_3 - x_2 &= 1 \\
x_2 - x_4 &= 1 \\
x_1 &\neq x_3 \\
x_2 &\neq x_4
\end{aligned}
$$



More technical details about MinCSP algorithms: PCCR workshop (Friday-Saturday)

Summary

- **CSP**: consistency-checking problem
- **MinCSP**: resolving *few* inconsistencies
- **Parameterized Complexity**: a natural framework for fixing *few* inconsistencies
- **Fine-Grained Analysis**: study the complexity of every fragments of the formalism

- Point Algebra ✅
- STP ✅
- 2-DTP (this work) ✅ New algorithm: equations and disequations

Future work

- Point Algebra ✅, STP ✅, 2-DTP ✅
- Is the 2-DTP criterion decidable ❓
- Temporal CSPs ❓ // relations first-order definable over $(\mathbb{Q}; <)$
- Other reasoning formalisms (e.g., spatial reasoning) ❓